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A local instructional theory for the guided reinvention of the group and isomorphism concepts

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ABSTRACT

In this paper I describe a local instructional theory for supporting the guided reinvention of the group and isomorphism concepts. This instructional theory takes the form of a sequence of key steps as students reinvent these fundamental group theoretic concepts beginning with an investigation of geometric symmetry. I describe these steps and frame them in terms of the theory of Realistic Mathematics Education. Each step of the local instructional theory is illustrated using samples of students' written work or discussion excerpts.

The research reported here is part of a larger project, Teaching Abstract Algebra for Understanding (TAAFU), focused on the creation of an innovative, research-based, inquiry-oriented curriculum for abstract algebra. Here we focus on the research and design work that supported the development of the *isomorphism* and *group* units of the TAAFU curriculum. This research fits within a growing body of research that is exploring the utility of the instructional design theory of Realistic Mathematics Education (RME) for supporting the learning of undergraduate mathematics. For example, there are ongoing RME-guided instructional design projects at the undergraduate level in the areas of geometry (Zandieh & Rasmussen, 2010), linear algebra (Wawro, Rasmussen, Zandieh, Sweeney, & Larson, 2012), and differential equations (Rasmussen & Kwon, 2007). Like these projects, my research aims both to contribute to the ongoing development of RME and to contribute knowledge about the learning and teaching of particular mathematical topics (groups and isomorphism).

Students' difficulties in abstract algebra courses have been well documented (e.g., Asiala, Dubinsky, Mathews, Morics, & Oktac, 1997; Hart, 1994; Leron, Hazzan, & Zazkis, 1995; Weber & Larsen, 2008). Students struggle not only with proving theorems in abstract algebra (Weber, 2001), but also with the level of abstraction (Hazzan, 1999) and complexity (Leron et al., 1995) of the fundamental concepts. The purpose of the local instructional theory (LIT) presented here is to support the design of instruction that addresses these difficulties by actively engaging the students in the reinvention of the concepts of group and isomorphism.

Freudenthal (1973) argued that groups should be introduced as systems of automorphisms of structures under composition. He suggested that when groups are introduced in this way, the group axioms could be verified conceptually rather than algorithmically. Consider the set of symmetries of an equilateral triangle:

- Closure: If one combines two symmetries of an equilateral triangle it is clear that this combination is also a symmetry of an equilateral triangle.
- Identity: The trivial symmetry (360° rotation) clearly acts as an identity element.

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- Inverses: A symmetry can clearly be 'undone' by reversing the transformation.
- Associativity: When performing three symmetry transformations in sequence, it clearly does not matter whether one imagines grouping the first pair or the second pair.
- In this sense, the group structure can be seen as a natural result of working with a system of automorphisms of a structure under composition.

Burn (1996) similarly argued for beginning instruction of group theory with an informal investigation of permutation and symmetry. However, Dubinsky, Dautermann, Leron, and Zazkis (1997) cautioned that, although from a mathematician's perspective group concepts may be visible in specific examples, it may still be difficult for a student to abstract those group concepts from the specific examples. The goal of the research and design process that produced the LIT presented here was to develop an instructional approach that would capitalize on the potential Freudenthal and Burn identified in the context of geometric symmetry while also attending to the concerns of Dubinsky et al.

A local instructional theory describes how a specific topic, such as isomorphism, could be taught in accordance with certain curriculum design principles. In the case of local instructional theories that are consistent with the theory of Realistic Mathematics Education, the primary principle is that the instruction should "comply with the intent to give the students the opportunity to reinvent mathematics" (Gravemeijer, 1998, p. 280). A local instructional theory differs from an instructional sequence in that its focus is not on the sequence itself, but on providing a rationale for the sequence. This rationale is in the form of a set of empirically and theoretically supported conjectures regarding how the mathematics will emerge.

In the following section, I will describe the theoretical perspective that supported the development of the LIT. Then I will describe the methods used to develop the LIT. Finally, I will present the LIT in two sections, one focused on the group concept and one focused on the isomorphism concept. The LIT will take the form of a sequence of key steps in terms of students' mathematical activity. Some of the steps represent key mathematical activities students engage in during the reinvention process, while other steps represent key ways of understanding (Harel, 2007) that are milestones in the development of the mathematical ideas. These steps will be framed in terms of the theoretical constructs that supported the creation of the LIT, and will be illustrated with examples of students' mathematical activity in the form of written work or discussion excerpts.

1. Theoretical perspective

The theoretical support for the research and design process was provided by the instructional design theory of Realistic Mathematics Education (RME). *Guided reinvention* is an overarching design heuristic of RME and is central to the process described here. Gravemeijer and Doorman (1999) explain that the idea of guided reinvention "is to allow learners to come to regard the knowledge that they acquire as their own private knowledge, knowledge for which they themselves are responsible" (p. 116).[1]

While guided reinvention provides a general heuristic for thinking about how one could (or should) support students' learning of mathematics, it does not provide specific guidelines for designing instruction. This type of support was provided by the *emergent models* heuristic. According to Gravemeijer (1999) emergent models are used in RME to promote the evolution of formal knowledge from students' informal knowledge. The idea is that the concept initially emerges as an informal and intuitive *model of* students' activity in an experientially real problem situation[2] and then the concept later evolves into a *model for* more formal activity. The concept is considered a *model-of* when an expert observer can describe the students' activity in terms of the concept. For example an observer may notice that, when students work informally with combinations of geometric transformations, their activity anticipates aspects of the algebraic structure of this group. The concept is considered to be a *model-for* when students can use the concept to support their reasoning in a new situation. For example, the algebraic structure of the group of geometric symmetries can serve as a model for analyzing various systems (e.g., the integers under addition) and developing a formal definition of group. In this way, the transition from model-of to model-for can be seen as a transition to more general mathematical activity. Note that the model is an overarching concept that can appear in various forms during its development. For example, the algebraic structure of the group of symmetries of an equilateral triangle is a model that can appear in the form of a list of symmetries, and operation table, or set of rules for manipulating symbols.

The instructional design efforts (and my analyses of the participating students' mathematical activity) were guided by Gravemeijer's (1999) description of the ingredients of a local instructional theory. In particular, I worked to:

- Discover student strategies and ways of thinking that anticipate the formal concepts.
- Identify design principles for instructional activities that could be used to *evoke* these strategies and ways of thinking.
- Identify design principles for instructional activities could be used to *leverage* these strategies and ways of thinking to support the development of the formal concepts.

[1] It should be noted that the guided reinvention heuristic does not imply that the students must reinvent the ideas without the assistance of a teacher (see Rasmussen & Marrongelle, 2006, for a detailed discussion of the role of the teacher in inquiry-oriented instruction).

[2] Experientially real problem situations are not limited to situations one would encounter in everyday life. Instead the term refers to "that which at a certain stage common sense experiences as real" (Freudenthal, 1991, p. 17).

The LIT that I describe in this paper is the culmination of the various insights of these three types that have resulted from analyses of students' mathematical activity. It describes a path by which students can reinvent the formal concepts of group and isomorphism beginning with informal activity in the context of geometric symmetry.

2. Research method

The research reported here consists of a series of design experiments (The Design-Based Research Collective, 2003). Each design experiment involved a series of cycles that involved designing tasks, engaging students with the tasks, analyzing the resulting mathematical activity, and then redesigning the tasks and designing new follow-up tasks. Analyses of data collected during each experiment informed the ongoing development of the emerging local instructional theory and the design of specific instructional tasks. Here we focus on the first three phases of the research and design cycle:

Phase 1: Small-scale design experiments. The first phase consisted of a series of three design experiments each conducted with a pair of undergraduate students. All of the students completed their participation in the design experiment *before* enrolling in their first abstract algebra course. The first pair of students consisted of a math major (and pre-service teacher) who had just received an "A" in a transition to proof course taught by the author, and a non-degree seeking student who had just received a "C" in the same course. The second pair consisted of two math majors who had both received an "A" in the transition course, and the third pair consisted of two math majors (both pre-service teachers) who had both received a "C" in the same transition course. The teacher/researcher met with each pair for seven or eight sessions, each lasting approximately 90 min. Each session was video-recorded and all of the students' written work was collected.

Fig. 1 illustrates Phase 1 of the research and design process. Ongoing analyses were conducted between sessions during each design experiment. The purpose of these analyses was to formulate and test conjectures about students' mathematical activity and to design new tasks to support or leverage this activity to support the development of the formal concepts of group and isomorphism. Further analysis was conducted between design experiments in order to refine the emerging local instructional theory and develop conjectures regarding how the students' most powerful ideas could be evoked and leveraged to support the development of the formal concepts.

A retrospective analysis of the data generated from all three small-scale design experiments resulted in an initial version of the LIT and a sequence of specific instructional tasks that fed into the subsequent phase of the research and design process. The retrospective analysis conducted after Phase 1 was carried out with the goal of adapting and refining the LIT to support instruction in the context of an undergraduate group theory course. This retrospective analysis involved multiple cycles of video analysis (Cobb & Whitenack, 1996; Lesh & Lehrer, 2000). The first cycle of the retrospective analysis was focused on

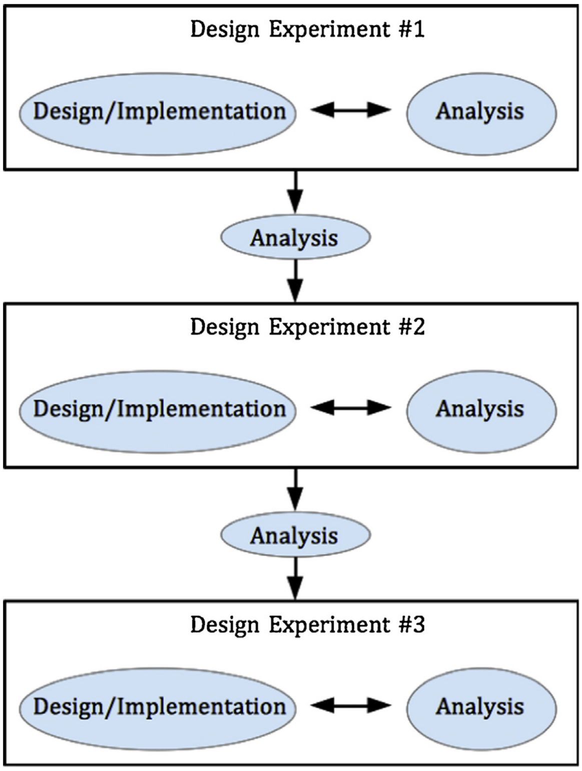

Fig. 1. Structure of Phase 1.

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Table 1
The four essential research and design events of Phase 2.

Term	Activity	Data collection
Summer 2004	Experimental teaching in group theory course	None
Fall 2004	Whole class teaching experiment	Classroom video
Summer 2005	Teaching experiment in algebra course for K-12 teachers	Classroom video
Fall 2007	Whole class teaching experiment w/mathematician as teacher	Debriefing/planning video

identifying student strategies and ways of thinking that anticipated the formal concepts. The second cycle of analysis was focused on constructing explanations for the emergence of the students' most productive and powerful ideas. The third cycle of analysis was similar, but was focused on identifying principles for leveraging the students' informal ideas in order to promote the development of the formal concepts.

Phase 2: Adaptation for regular classroom setting. The instructional theory and instructional sequence that resulted from Phase 1 of the research and design process was adapted and refined for use in a classroom setting through a combination of *experimental teaching* (Steffe and Thompson, 2000) and whole-class *teaching experiments* (Cobb, 2000). First, I taught an undergraduate group theory course during a summer session in order to experiment informally with using the LIT to develop instructional materials for use in a classroom setting. Then I conducted a whole-class teaching experiment the following term in an undergraduate group theory course, video-recording every class session. This was followed by a whole-class teaching experiment conducted in a special algebra course designed for K-12 teachers. Finally, a whole-class teaching experiment was conducted in collaboration with a mathematician teaching an undergraduate abstract algebra course. Methodologically, the primary difference between the experimental teaching and the teaching experiments is that, like the design experiments that comprised the first phase of the project, the teaching experiments featured ongoing analyses between class sessions. As with the Phase 1 design experiments, the purpose of these analyses was to develop and test conjectures regarding how to evoke and leverage students' informal ideas in order to support the development of the formal concepts of group and isomorphism. Table 1 delineates the most essential components of Phase 2 of the research and design process.[3]

The ongoing and retrospective analyses of the video data generated during Phase 2 were primarily focused on accounting for challenges and opportunities afforded by the classroom setting. For example, because the LIT called for students to design their own symbols (e.g., for the symmetries of an equilateral triangle), it needed to be refined to include strategies for developing a single set of symbols that could be shared by the classroom community.

3. The local instructional theory

The purpose of this paper will not be to describe in full detail the actual reinvention process as it unfolded in any one of these studies.[4] Here the goal will be to present and illustrate the LIT, which can be seen as an abstraction of the various reinvention processes that were observed during the course of the research and design process. To maximize the coherence of the presentation, the LIT will be illustrated primarily using data drawn from the first small-scale design experiment conducted with two students who will be referred to as Jessica and Sandra. Additional data excerpts (including SMART Board captures) will be drawn from whole-class implementations of the TAAFU curriculum in order to illustrate aspects of the LIT that were added or refined substantively during Phase 2 of the research and design process.

3.1. LIT part 1: the group concept

The first part of the LIT is focused on the group concept. It can be best described as a sequence of steps in terms of students' progressive mathematical activity. The reinvention process begins in the context of the symmetries of a geometric figure as students identify, describe, and symbolize the set of symmetries of (usually) an equilateral triangle. The group structure begins to emerge as a model-of the students' mathematical activity as they begin to analyze combinations of pairs of symmetries. Perhaps the most important shift in the students' activity occurs when they transition from analyzing combinations geometrically to calculating combinations algebraically using a set of rules that they develop.

The rules that the students develop in order to compute combinations of symmetries include axioms featured in the definition of group as well as relations specific to the symmetries of the geometric figure. This set of rules will typically consist of more than one version of the dihedral relation and may include other "unnecessary" rules as well. Students then reduce their list of rules to a minimal set of rules needed to completely determine the operation table for combining pairs of symmetries. Note that at this point the students have transitioned from mathematizing the geometric context to mathematizing their own mathematical activity (their calculus for computing combinations of symmetries). The reinvention

[3] The resulting curriculum was implemented a number of additional times by various instructors and these implementations informed the refinement of the LIT in minor ways, however the four events listed in Table 1 were the most significant events of Phase 2.

[4] For a detailed description of the reinvention process that unfolded during the first three design experiments and a discussion of how the initial LIT emerged as a result of these experiments, see Larsen (2009) or Larsen (2004).

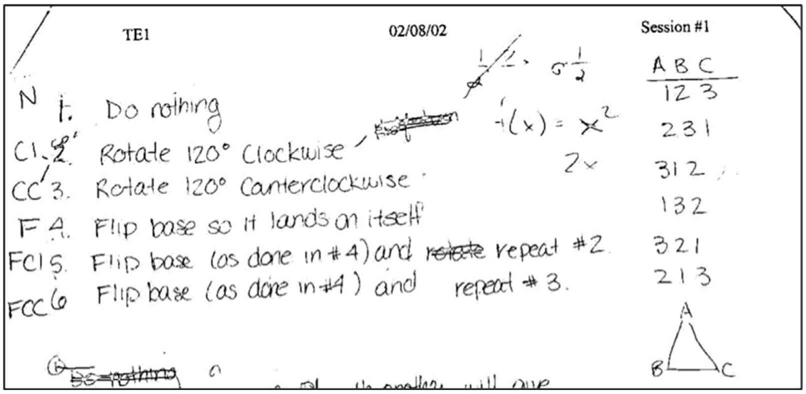

Fig. 2. The symmetries of an equilateral triangle.

process concludes with students analyzing other systems (groups) and defining the group concept in terms of the properties shared by these systems.

Step 1(A): Identifying and symbolizing the symmetries of a specific geometric figure.

The starting point context for the reinvention of the group concept is the set of symmetries of a geometric figure. The data samples used for illustration in this paper are drawn from students' work with the symmetries of an equilateral triangle. The first step in the reinvention process is for students to identify, describe, and create symbols for the symmetries. This requires the students to establish what it means for two symmetries to be equivalent. Typically students agree that two symmetries are equivalent if they have the same effect on the triangle in terms of how the vertices are permuted. Fig. 2 was produced by Jessica and Sandra, the participants in the first small-scale design experiment.

Notice that the last two symmetries in the list are symbolized using a combination of a rotation symbol (*CC* or *CL*) and a reflection symbol (*F*). Jessica and Sandra referred to these two symmetries as 'compound moves'. While all of the pairs who participated in the small-scaled design experiments used this kind of approach, subsequent work in classroom settings has revealed that it is much more common for students to express all of the reflections as single motions.[5]

Step 1(B): Negotiating a shared set of symbols that includes compound moves.

As the research and design process moved to the classroom setting, it became important to include a sub-step of the LIT that was focused on negotiating a symbol set that could be shared by the classroom community. Retrospective analyses of the small-scale design experiments revealed the importance of the inclusion of so-called compound moves in order to support a subsequent crucial step of the reinvention process. This will be discussed in detail below when I describe Step 3 of the LIT. Here, we will focus on the strategy for supporting the negotiation of a shared set of symbols that includes compound moves.

Essentially the approach taken by the LIT is to use the need to address one design problem as a vehicle for addressing a second design problem. Motivated by the need for the classroom community to establish a single shared way of representing the symmetries in order to facilitate communication during whole class discussions, the LIT calls for the students to represent all six symmetries using the two basic symbols F and R (where F represents the reflection across the vertical axis of symmetry and R represents a 120° clockwise rotation). As students share their responses, a collection of different expressions for each symmetry emerges. These provide options to be considered in a whole-class negotiation to select a set of common symbols to be used by the classroom community. Note that while the students have considerable freedom in selecting their symbol set, two of the reflections require the use of both F's and R's. This ensures that the final symbol set will include compound moves.

Fig. 3 is a SMART Board capture from a whole-class discussion in which a common set of symbols was selected. The circled expressions were selected by the classroom community to be the standard set of symbols for the symmetries of an equilateral triangle.[6]

Step 2: Combining pairs of symmetries.

The activity of identifying and symbolizing the symmetries of an equilateral triangle results in the listing of a set. However, it is not until the students start performing an operation on these symmetries that their activity can begin to be seen as anticipating the group structure of this system. The seeds are planted for this in Step 1(B) when the students express the symmetries in terms of two generators (F and R). The operation of combining symmetries becomes the focus of the students' activity in the second step of the LIT.

Students are tasked with considering each pair of symmetries and determining which of the six symmetries is equivalent to this combination. Often students begin by (apparently) arbitrarily selecting pairs to analyze. However, they soon begin

5 Most abstract algebra texts (e.g. Durbin, 2008) also express the reflection symmetries of regular polygons as single motions.
6 This SMART Board capture does not include the identity symmetry. The class selected the symbol "*I*" to represent the identity symmetry.

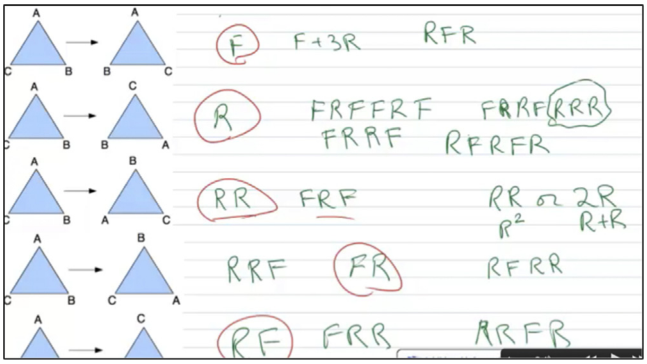

Fig. 3. Class notation for symmetries in terms of generators.

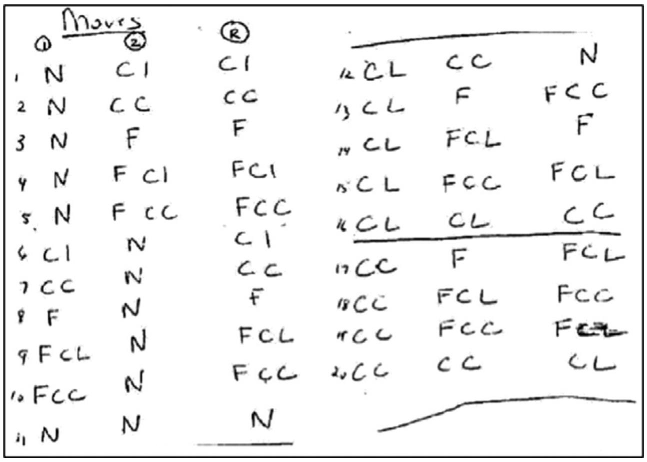

Fig. 4. Structured list of symmetry combinations.

to record their results more systematically. For example, students may generate a structured list like the one that appears in Fig. 4.[7]

Organized lists of the type shown in Fig. 4 can be further structured to form operation tables like the one seen in Fig. 5. This transition to using operation tables can be motivated by a need to know when the task is complete, but can also driven by students' prior experiences with such tables. In the event that students do not come up with the idea of organizing their results in the form of a table, the instructor can introduce this idea to provide students with more powerful method of recording their activity. This is an example of what Rasmussen and Marrongelle (2006) refer to as a transformational record, a pedagogical strategy (inspired by the emergent models heuristic) designed to build on students' ideas by introducing more powerful ways of representing their mathematical activity.

Operation tables have an important role in supporting the model-of/model-for transition because, while they are initially simply records of the students' informal activity, they transition to serving as tools for supporting more formal mathematical activity including the construction of deductive proofs. For instance, Weber and Larsen (2008) describe a students' use of an operation table to create a general proof that the identity element of a group is unique.

This phase of the reinvention process represents the initial emergence of the group concept as a model of the students' informal activity. While an expert may recognize the operation table in Fig. 5 as a group table, the students producing such a table may not be explicitly aware of the role of the group axioms in their work.

However, in the initial design experiments, I observed students engaging in activity that was more explicitly group theoretic (and will be the focus of the next step of the LIT). The students spontaneously began using shortcuts to determine some of the symmetries without manipulating a triangle or drawing figures. For example, in the following excerpt we

[7] Students typically write combinations of symmetries in left to right order rather than using the right to left order that is standard convention for composition notation.

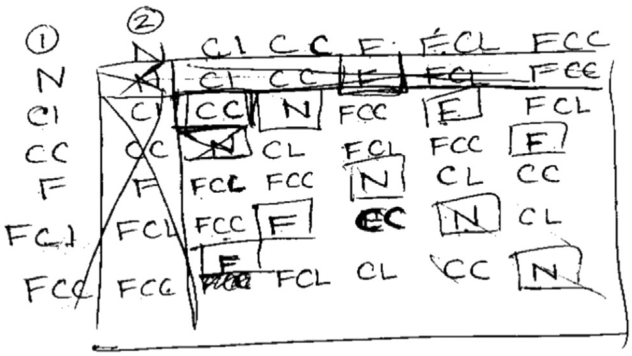

Fig. 5. Operation table for combining symmetries.

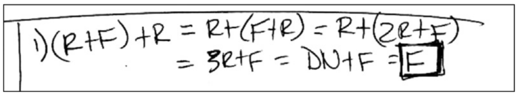

Fig. 6. Student calculation of a combination of two symmetries.

see Jessica observing that any combination involving the "do nothing" move can be computed by simply disregarding this move.

Jessica:	So if we do "do nothing" and one of these other ones, it's gonna be the same thing.
Sandra:	Right.
Jessica:	So should we address that?
Teacher/Researcher:	Yeah, you could write that down.
Sandra:	That's a very good observation. I like that.

The students in the initial design experiment went on to develop a sophisticated system for calculating combinations of symmetries symbolically using rules that they verified intuitively or by manipulating a physical triangle. In my analysis, I identified this strategy as particularly powerful for supporting students' reinvention of the group concept. Thus the development of such a system became perhaps the key step in the resulting LIT. In the next section, I will discuss this step of the LIT in detail and the role of compound movies in evoking it. The three sections that follow will describe how this strategy can be leveraged to support the development of the formal group concept.

Step 3: Developing a calculus for computing combinations of symmetries.

The various cycles of research and design have revealed that some students will spontaneously develop a sophisticated scheme for computing combinations of symmetries. This was the case with Jessica and Sandra during the first design experiment and the process is detailed in Larsen (2009). However, this is not always the case. What is typical is that students will use *some* basic shortcuts as they create their operation tables. In particular, it is very common for students to realize that one of the moves has no effect when included in a combination (identity) and that every symmetry appears exactly once in each row and column of the operation table (typically called the Sudoku property). The LIT is designed to leverage students' use of basic observations like these by asking students if they can generate enough rules of this type to compute all of the combinations of symmetries without physically manipulating the triangle. Students are asked to record a list of emerging rules/observations and use them to compute all 36 calculations. For example, Jessica and Sandra carried out the calculation shown in Fig. 6 by using associativity, a dihedral relation ($F+R=2R+F$), the relation $3R=DN$ ($DN=$"do nothing"), and the identity property.

This example can be used to illustrate the crucial role that compound moves play in evoking the strategy of performing rule-based calculations. While it is likely that the use of the identity property could emerge regardless of the symbolic system, it is difficult to imagine students using regrouping and substitution to compute pairs of symmetries that do not involve compound moves. This is discussed in great detail in Larsen (2009). Here I will simply observe that the concatenation of symbols like F_1 and F_2 does not suggest the possibility of a rule-based calculation in the way that the concatenation of the symbols F and $F+R$ does.[8] The symbolic expression $F+(F+R)$ for example, invites regrouping the F's and then ignoring them because performing a flip twice is the same as doing nothing.

Before discussing the importance of the strategy of rule-based calculations in the reinvention process, it is worth noting that not all of the group axioms emerge explicitly as rules and when they do, they do not necessarily emerge in the form that they appear in the definition of group. This is most significant in the case of the inverse and associativity axioms. While students typically do use inverses in their work (this can be seen in the calculation shown in Fig. 6) as a way to simplify

[8] Sometimes students represent the three reflection symmetries of an equilateral triangle using symbols like F_1, F_2, and F_3.

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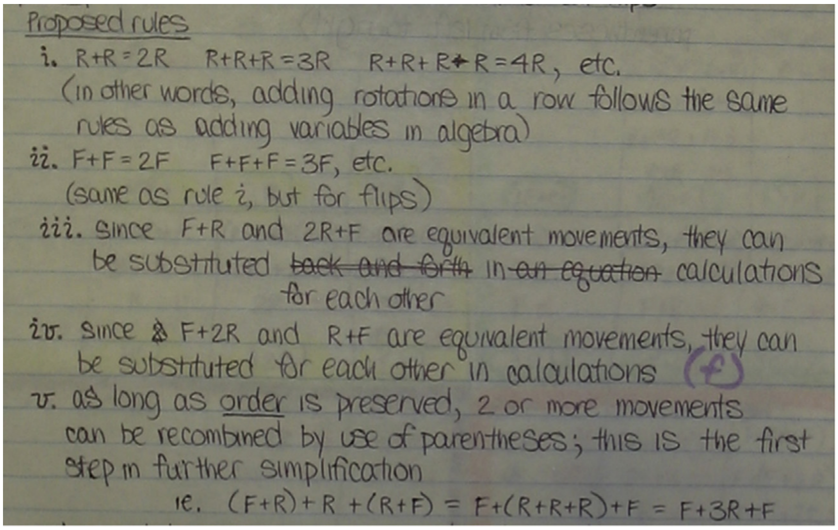

Fig. 7. List of rules for combining symmetries.

expressions, it is rare for students to include a rule stating that every symmetry has an inverse. Step 5 of the LIT directly addresses this issue.

The issue with the associative property is a bit different. Students freely make use of grouping, but may not express this process using bracketing symbols and may not be aware of the significance of this property because it seems to be clear that it does not matter how the elements are grouped. Further, students sometimes become confused by the fact that associativity seems to suggest that 'order does not matter' while the lack of commutativity of this system suggests that 'order does matter'. These and other issues related to the associative property can be explained by the fact that the operation of combining symmetries can be thought of as a string of actions rather than as a binary operation[9] (Larsen, 2010). In terms of implementing a curriculum sequence based on the LIT presented here, the pedagogical challenge is to make sure that students explicitly attend to their use of regrouping so that it is included in their emerging list of rules.

We can see the significance of the emergence of rule-based calculations in terms of the emergent models construct in several ways. First, we can see (from an expert's perspective) that much more of the group concept is explicit in the students work. In particular, the students are explicitly referring to and using properties included in the definition of group. Second, the students' mathematical activity is more general. The students are relying much less on the particulars of the specific context and their activity is focused on their emerging system for manipulating their symbols rather than the geometric context. Third, this emerging calculus for combining symmetries represents a new mathematical reality for the students that can in turn be further mathematized to emphasize the algebraic (group theoretic) structure of this system of geometric transformations. This further mathematization is the focus of the next step of the LIT in which the students engage in axiomatizing the system of symmetries by making deductively supported decisions about whether their system of rules is complete and which of their rules should be considered to be essential.

Step 4(A): Axiomatizing the set of rules (Part 1: efficiency).

After the students develop a system of rules for computing combinations of symmetries algebraically, this system can be further mathematized by determining a minimal set of rules for calculating all 36 combinations of two symmetries. Such a minimal set of rules must include some version of the identity property, the associative property, and some relations specific to the symmetries of an equilateral triangle. The list of rules that students generate initially will include rules that, while helpful in computing the full set of combinations, are redundant in the sense that the task could actually be completed without using them. Redundant rules typically include multiple versions of the dihedral relation. For example, in Fig. 7 we see two different dihedral relations (rules iii and iv).

Students can test the redundancy of a give rule by attempting to prove it using the other rules. This supports the students in further mathematizing their system by eliminating some of the rules and awarding them status as theorems. Additionally, this process provides an opportunity for students to use their rules in a more sophisticated way (as steps in deductive proofs). In this sense, symbolic expressions that emerged originally as a record of the students' activity manipulating a triangle become tools in deductive arguments. For example, the observation that a combination of F and R is equivalent to the combination R^2F would typically emerge during Step 1(B) of the LIT as in the form of two different ways to represent one of the three reflection symmetries. This observation could then be used during Step 3 to compute calculations of the

[9] The view of the operation as a string of actions is actually a very powerful view akin to the use of group actions in advanced group theory. The operation becomes formalized in the standard way as a binary operation at the end of the LIT when a definition of group is formulated.

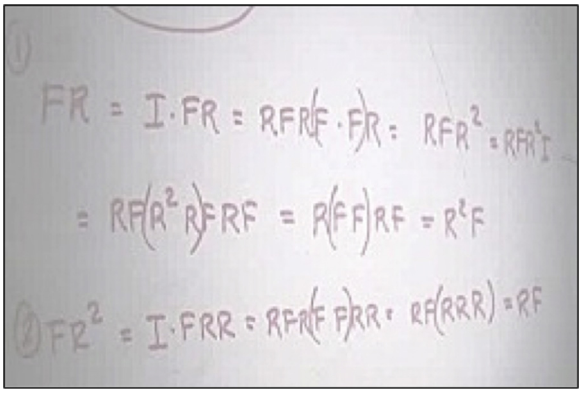

Fig. 8. Classroom proofs of "unnecessary" rules.

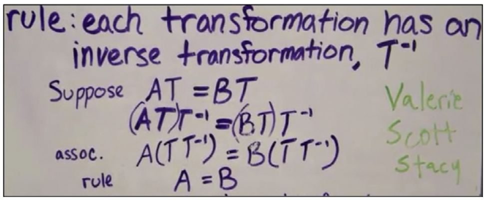

Fig. 9. Inverse property introduced and used to prove part of the Sudoku property.

type shown in Fig. 6. Finally, in this step (Step 4) this kind of rule is used in deductive arguments. For example, in Fig. 8 we see proofs of the rules $FR = R^2F$ and $FR^2 = RF$ using the group axioms and the rules: $FF = R^3 = I$, and $RFRF = I$.

This process of reducing the set of rules to an essential set serves the purpose of producing an efficient description of the algebraic structure of the set of symmetries of an equilateral triangle. However, the system that results from this activity will typically not be complete. As previously noted, students use inverses frequently in performing calculations, but they will not commonly include the existence of inverses in their list of rules. This is a rule that will need to make an appearance in the students' final formulation of the group concept. The next (sub)step in the process involves leveraging the common student assumption that the operation table satisfies the "Sudoku property" to elicit an explicit statement of the existence of inverses and motivate its inclusion in the basic set of rules for this system.

Step 4(B): Axiomatizing the set of rules (Part 2: completeness).

As noted earlier, students typically make the observation that each element appears exactly once in each row and column of the operation table for the symmetries of an equilateral triangle. Students commonly refer to this as the "Sudoku property". The next phase in the reinvention process is launched by asking students whether they could prove the Sudoku property using only their minimal set of rules. Note that this property *can* be proved by the minimal set of rules, because these rules can be used to calculate all 36 combinations of symmetries. The idea then, is to be able to prove the Sudoku property *efficiently* without having to do all these calculations. The need to produce such an efficient deductive argument motivates the need to complete the system by including a rule regarding existence of inverses.

The task of proving the Sudoku property can be decomposed into two separate tasks. Students can be asked to first consider the conjecture that each element can occur *at most once* in each row of the table, and then consider the conjecture that each element must occur *at least once* in each row of the table. Fig. 9 shows a student's proof that each element can occur at most once in each row of the table. Students typically express the situation in which an element appears twice in a row algebraically. Then the existence of inverses emerges as a way to justify the cancelation needed to complete the argument.[10] This is an instance in which the instructional design capitalizes on the *systematizing* role of proof (De Villiers, 1990) because the inclusion of the inverse axiom is motivated by the desire to create a system of axioms that is sufficiently powerful to efficiently prove properties that are already apparent to the students.

Notice that in addition to motivating the completion of the students' set of rules to include existence of inverses, these two tasks also anticipate two important basic theorems from group theory: the cancelation law and the existence of solutions to equations of the form $ax = b$. In the TAAFU curriculum that has been developed based on the LIT presented here, one of the first tasks after the definition of group has been formulated is to state and prove these two theorems rigorously.

[10] Similarly, the existence of inverses can be used to show that each element occurs at least once because they can be used to show that equations of the form $ax = b$ have solutions.

The axiomatizing step of the reinvention process is an important one in terms of the transition from model-of to model-for. There is a shift in the role of the set of rules from being an efficient way to describe the system to providing the foundation for deductive proofs. The rules become a framework for determining what must be true of a system that satisfies these rules. In particular, the identity property, the associative property, and the inverse property can be seen as a foundational set of rules that imply both the existence of solutions to equations and the validity of cancelation. With the existence of inverses established as part of the essential rule set for the symmetries of an equilateral triangle, all of the group axioms are included. The next step of the reinvention process is to transition to using this rule set as a framework for analyzing other systems.

Step 5: The system of rules as a model-for reasoning about other contexts.

The next phase of the reinvention process involves analyzing different contexts, involving a set and an operation of some sort, to determine whether they satisfy a similar set of rules. Working with similar systems, like the symmetries of a non-square rectangle, helps to emphasize the difference in generality between rules like the identity property and rules like the dihedral relation. Working with less similar systems, like the integers under addition, helps to broaden the students' view of what kinds of objects can comprise the system (e.g., it does not have to consist of a finite set of transformations). This step of the LIT directly addresses the concerns of Dubinsky et al. (1997) regarding the difference between an informal understanding of a specific group and a formal understanding of the group concept.

This phase continues the model-of/model-for transition. The students' strategies and ways of operating in the context of the equilateral triangle symmetries are now used as tools to analyze new systems. The system of rules provides a framework for determining to what extent these new systems work the same way as the symmetries of an equilateral triangle. It also represents a shift in terms of generality because the students are moving away from the equilateral triangle context. While the students' activity may refer back to the triangle context as they make comparisons, their activity is general in the sense that it is for the most part situated in the new contexts. Furthermore, it is the mathematized version of the starting point context (in the form of the students' minimal set of rules) that is the focus of these comparisons. This transition to more general activity continues in the 'final' phase of the reinvention process in which a definition of group is formulated as a way to characterize systems that exhibit the core rule set shared by the various systems the students have analyzed.

Step 6: Formulating a definition of group.

After analyzing a number of different systems (selected by the instructor, because from an expert's view they are examples of groups) the identity, associative, and inverse properties will emerge as the set of common properties. The term group can be introduced, by the instructor, to denote systems satisfying these properties. It can be challenging for students to articulate what exactly comprises such a system. Typically, it is necessary to ask students what they need to have (a set and an operation[11]) in terms of mathematical objects for these rules to make sense. Unsurprisingly, students may struggle with precisely formulating a definition using appropriate quantification structures (see Dubinsky & Yiparki, 2000, for more on the challenge of understanding quantification).

While this step concludes the first part of the LIT and culminates with the formulation of a formal definition of group, the group concept continues to develop in two ways as students engage with the TAAFU curriculum. First, the transition to a model-for continues as students use their new definition of group to prove various theorems (including the cancelation law) and to reinvent new concepts (e.g., isomorphism and quotient group). Second, the notion of an abstract group emerges with the reinvention of the isomorphism concept in that isomorphism makes it possible for different groups (e.g., the symmetries of an equilateral triangle and the permutations of a set of three elements) to be seen as instances of the same abstract group.

3.2. LIT part 2: the isomorphism concept

As with the group concept, the LIT for the isomorphism concept draws heavily on the emergent models construct. Isomorphism emerges first as a model-of the students' informal activity as they determine whether a given "mystery" table could be an operation table for the symmetries of an equilateral triangle. In order for the emerging isomorphism concept to support more formal reasoning, aspects of the students' activity need to become more explicit. Specifically the students' implicit respect of the operation (formally the homomorphism property) needs to become explicit. When the students are able to articulate what properties a correspondence between groups needs to possess (in order to demonstrate that two groups are essentially the same) these properties can be formulated into a definition of isomorphism. This definition can then be used as a tool for more formal reasoning (including proving that some pairs of groups that seem very different are equivalent as groups).

Step 1: Evoking and leveraging a naïve view of isomorphism.

The reinvention of the isomorphism concept takes as a starting point the students' earlier experiences developing their own symbols for the symmetries of an equilateral triangle. Since the students explicitly made decisions regarding what symbols the class would use to represent these symmetries, they are aware that different choices could have been made. Thus, they go into the process of reinventing isomorphism with the idea that the appearance of a group could vary. Note that this idea does not capture, even informally, the full notion of isomorphism in which groups whose differences go beyond

[11] Students typically define the operation to be a process that takes in two group elements and produces another group element (perhaps formalized as a function). Notice that this definition builds the closure axiom into the definition of operation.

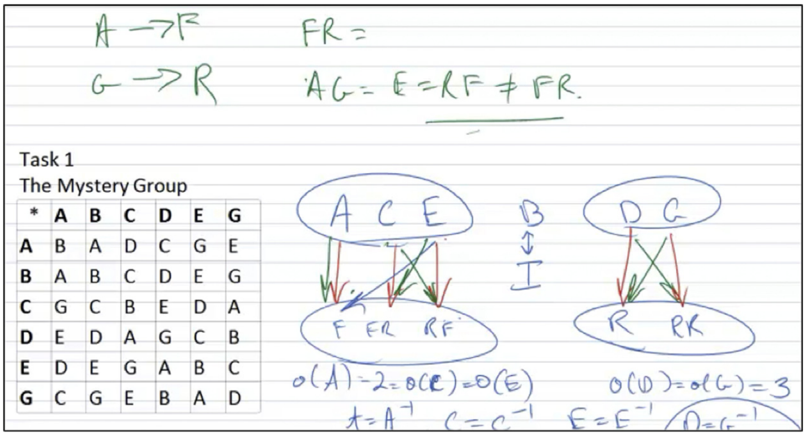

Fig. 10. SMART Board capture of the mystery table task.

surface features can still be considered to be equivalent (e.g., the positive real numbers under multiplication and the real numbers under addition). Rather, the idea is simply that the group of symmetries of a triangle could be represented using a different set of symbols. This is what Leron et al. (1995) refer to as a naïve view of isomorphism. However, the properties that constrain students' activity as they attempt to determine whether two groups are the same except for their symbol set are in fact sufficient for formulating the formal definition of isomorphism. The approach taken by the LIT is to evoke this naïve view of isomorphism and them support students in further mathematizing their activity in order to develop the formal concept.

Students' awareness that the symmetries of a triangle could be represented differently is leveraged to evoke the naïve view of isomorphism by asking them to consider a mystery table that represents a group isomorphic to the group of symmetries of an equilateral triangle. Such a table appears in the SMART Board capture shown in Fig. 10. Students are asked whether the mystery table could be the group of symmetries of an equilateral triangle. From the students' perspective, the question is whether this table could be an artifact of another individual's work with the symmetries of an equilateral triangle.

Students address this question by trying to determine which symmetry each of these new symbols could be representing. Typically, they will begin by identifying the identity element of the mystery table. In the table shown in Fig. 10, this is **B**. It is common that the students' strategy is to then identify the other elements that are self-inverses, because these are the ones that must represent the flip symmetries. In the table shown in Fig. 10, these are elements **A**, **C**, and **E**. Armed with the idea that **B** must be the identity, and **A**, **C**, and **E** must represent flip symmetries, the students use various strategies for constructing and testing a complete correspondence between this new symbol set and their set of symbols for the symmetries. Some students simply arbitrarily assign the three flip symmetries to the symbols **A**, **C**, and **E**, and then arbitrarily assign the two rotation symmetries the remaining symbols in the mystery table. In Fig. 10, these symbols are **D** and **F**. Other students implicitly make use of the homomorphism property to make their assignments. For example, they may assign some elements arbitrarily and then build partial operation tables or perform calculations (see top of Fig. 10) to determine how to assign the remaining symbols.

The SMART Board capture shown in Fig. 10 actually shows (using colored arrows) two different correspondences that work. Students then can test these conjectured correspondences by some combination of changing the names of the symbols (in either the mystery table or the operation table for the symmetries of an equilateral triangle) and rearranging the tables in order to verify that mystery table really does work like the original operation table for the triangle symmetries under this correspondence.

In terms of the emergent models heuristic, this phase of the reinvention process can be seen as the emergence of isomorphism as a model-of the students' informal mathematical activity. The students' ways of attacking the problem can be seen as anticipating the conditions outlined in the formal definition of isomorphism (e.g., the existence of a bijection that works in the sense that corresponding pairs of elements have corresponding products). Before this model can evolve into a model for more formal reasoning, these aspects of the students' ways of operating need to become more explicit.

Step 2(A): Emergence of the homomorphism property (Analyzing a non-homomorphism).

It is quite common for students to respect the operation when developing correspondences between symbols in the mystery table and the symmetries of an equilateral triangle. And in fact, students will typically discover some correspondences that do not work even though all self-inverses correspond to self-inverses. However, the students' use of the homomorphism property is usually largely or completely implicit. The focus of this (sub)step of the LIT is on raising the students' awareness of this aspect of their mathematical activity. First students are asked to explain why a one-to-one correspondence that seems viable (the identities are in correspondence and all self-inverses correspond to self-inverses)

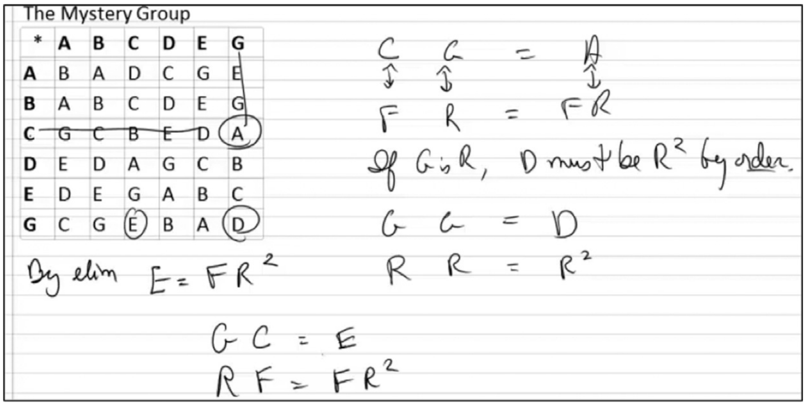

Fig. 11. Using the homomorphism property to complete a correspondence.

does not work. In the excerpt below, Jessica and Sandra, explain why such a correspondence does not work in the context of a mystery table somewhat different from the one shown in Fig. 10 (the elements were represented by the numbers 1–6).

SL: So what is the problem, so tell me the problem Sandra.
Sandra: So you gave 2 a name of flip and you gave R a name of 4 but then FR is a name of 3, which is a problem.
SL: What do you think it should be?
Sandra: It should be um. . .
Jessica: 6.
Sandra: 6, yeah.
SL: Because?
Jessica: 2 dot 4 is 6

In this case, Sandra has identified a product in the mystery table, 2·4, that produced an answer, 6, that did not correspond to the result, FR, of the corresponding product of symmetries, FR (FR corresponded to 3 not 6). Note that this activity of searching for a place where the correspondence breaks is explicitly focused on the operations and how well they correspond, whereas the students' explicit attention during the initial mystery table task was more on the attributes of individual elements. However, this property is still somewhat implicit and really appears more as an artifact of the students' activity of searching for a place where correspondence breaks down than as a tool for supporting that activity. The next (sub)step in the LIT features an important shift to *explicitly using* this property as a tool for constructing a correspondence that works.

Step 2(B): Emergence of the homomorphism property (Completing a homomorphism).

The next (sub)step of the LIT involves the emergence of the homomorphism property as an explicit tool for creating and reasoning about correspondences between groups. The instructional strategy is to engage students in completing a correspondence that has been defined for a set of generators of the group. Again the focus is on a correspondence between the mystery group and the group of symmetries of an equilateral triangle. Recall that the early stages of the LIT involved expressing all six symmetries of an equilateral triangle in terms of the generators F and R. This (sub)step engages students in completing a correspondence that has been defined for F and R. Fig. 11 is a SMART Board screen capture illustrating students' work extending the correspondence defined by $C \leftrightarrow F$ and $G \leftrightarrow R$. The calculations found in the top right corner illustrate the explicit use of the homomorphism property to determine where to map an element.

Students' engagement in this activity can be seen as supporting the model-of/model-for transition because they are now using (their awareness of) the homomorphism property as a tool for constructing a one-to-one correspondence that works. This supports the transition of the isomorphism concept from model-of to model-for in the sense that it anticipates the use of the definition of isomorphism to construct mappings in order to prove groups are isomorphic. The next phase of the LIT continues to promote this transition by recasting this awareness of the homomorphism property in the form of an algebraically represented condition that can be tested for any mapping between groups.

Step 3: Formulating an algebraic expression of the homomorphism property.

Once the homomorphism property becomes an explicit part of the students' informal activity, the remaining steps in the reinvention process involve formalizing the students' activity in a definition of isomorphism. An important step in this process is representing the homomorphism property as an algebraic expression based on the students' process for creating a correspondence that works. For example, in Fig. 12, we see this process formulated as an "if-then" statement that is equivalent to the standard statement that appears in the formal definition of isomorphism.

Students' further efforts to formulate a definition can be supported by calling their attention to the connections between the function concept and the process they used when working with the mystery table tasks. Notice that in Figs. 10–12 we see the students' work represented using arrows in a way that is evocative of the mapping view of function. Students can be engaged in discussing the idea that these correspondences are functions and then encouraged to use function notation to formulate the homomorphism property. In Fig. 13 we see a student formulation that uses function notation but retains the implication structure seen in Fig. 12 rather than the more efficient formulation, $f(i*j)=f(i)*f(j)$, found in textbook definitions.

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$\forall\, g,i,f \in G, \quad h,k,n \in H$

If $g \to h$, $i \to k$, $gi = f$

and $hk = n$ then $f \to n$

Fig. 12. Homomorphism property in mapping notation.

$i \cdot j = k \Rightarrow f(i) * f(j) = f(k)$

Fig. 13. Homomorphism property expressed as an implication using function notation.

This formulation of the homomorphism property as an algebraic statement (as opposed to an informal checking procedure) contributes in an important way to the model-of/model-for transition of the isomorphism concept. Specifically this transforms the students' informal (and largely ad hoc) activity into an explicit tool expressed in a general form that can be utilized in more advanced mathematical activity. More immediately, it makes it possible for students to formulate a definition of isomorphism.

Step 4: Formulating a definition of isomorphism.

The "final" step of the process of reinventing the isomorphism concept is to formulate a definition of isomorphism. Students incorporate the homomorphism property into a definition expressed in terms of (the existence of) a bijective mapping. Students typically are aware that a correspondence that "works" between the groups must be bijective (as indicated by their use of two directional arrows when they describe their correspondences), but expressing the definition using an existential quantifier can be challenging. Fig. 14 shows a student's definition of isomorphism.

This step concludes the second part of the LIT and culminates with the formulation of a formal definition of isomorphism. However, the isomorphism concept continues to develop as students engage with the TAAFU curriculum and work with this concept deductively. Specifically, the transition to a model-for continues as students use their new definition of group to prove various theorems, including conjectures that emerged during their work with the mystery group task (e.g., an isomorphism must map self-inverses to self-inverses).

4. Discussion

In this paper, I have presented a local instructional theory for supporting the guided reinvention of the group and isomorphism concepts. This local instructional theory takes as its point of departure the context of the symmetries of a geometric figure as students identify, describe, and symbolize the set of symmetries of an equilateral triangle. The group structure begins to emerge as a model-of the students' mathematical activity as they begin to analyze combinations of pairs of symmetries. Perhaps the most important shift in the students' activity occurs when they transition from analyzing combinations geometrically to calculating combinations algebraically using a set of rules that they develop. The rules that the students develop in order to compute combinations of symmetries include axioms featured in the definition of group as well as relations specific to the symmetries of the geometric figure. Students then further mathematize (systematize) their informal activity by reducing this list to include only essential rules and by expanding it to include the existence of inverses. The reinvention process then shifts to more general activity as students analyze other systems and define the group concept in terms of the properties shared by these systems.

Once the concept of group is established, the LIT then shifts to leveraging students' awareness that the group of symmetries of a triangle could have been represented by a different symbol set to develop the isomorphism concept. This concept emerges first as a model-of the students' informal activity as they determine whether a given mystery table could be an operation table for the symmetries of an equilateral triangle. This activity includes the creation of correspondences between the elements of the mystery table and the symmetries of an equilateral triangle. Students then mathematize this informal activity by investigating reasonable correspondences that do not work. This increases the students' awareness of the need to respect the operations (homomorphism property) that has previously been implicit in their work. This property becomes more explicit when students complete defining a working correspondence that has been defined in terms of the generators of

$\exists$ bijective $f: G \to H$

$\forall a,b \in G$, $f(a \cdot b) = f(a) * f(b)$

Fig. 14. Student definition of isomorphism.

the group of symmetries. The students then use function notation to formulate this property symbolically and incorporate it (along with the condition that the correspondence be bijective) into a formal definition isomorphism. This definition can then be used as a tool for more formal reasoning (including proving that some pairs of groups that seem very different are equivalent as groups).

Our research suggests that Burn (1996) and Freudenthal (1973) were right to see promise in symmetry as a fundamental concept of group theory, while Dubinsky et al. (1997) were correct in pointing out that it may be difficult for a student to abstract formal group concepts from the specific examples. The LIT presented here describes a trajectory by which students can navigate the distance between this concrete example and the general group concept.

The primary purpose of a local instructional theory is to support the design of an instructional sequence that is appropriate for a given instructional context. As part of the TAAFU project, three different instructional sequences have been developed based on this LIT: one for a community college "bridge" course, one for an algebra course for K-12 teachers, and one for use in an undergraduate abstract algebra course. We have observed consistent regularities across various instructional contexts in terms of students' mathematical activity. This suggests that the theory is robust enough to support successful adaptations of the instructional sequences. The curriculum materials for the undergraduate abstract algebra course have been integrated with a set of interactive instructor support materials (Lockwood, Johnson, & Larsen, 2013) and can be found online at http://www.web.pdx.edu/~slarsen/TAAFU/home.php.

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