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# A local instructional theory for the guided reinvention of the quotient group concept

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### ABSTRACT

In this paper we describe a local instructional theory for supporting the guided reinvention of the quotient group concept. This local instructional theory takes the form of a sequence of key steps in the process of reinventing the quotient group concept. We describe these steps and frame them in terms of the theory of Realistic Mathematics Education. Each step of the local instructional theory is illustrated using example instructional tasks and either samples of students' written work or excerpts of discussions.

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The research reported here is part of a larger project, Teaching Abstract Algebra for Understanding (TAAFU), focused on the creation of an innovative, research-based, inquiry-oriented curriculum for abstract algebra. Here we focus on the research and design work that supported the development of the quotient group unit of the curriculum. Our primary purpose is to describe the resulting *local instructional theory* (Gravemeijer, 1998) for supporting the guided reinvention of the quotient group concept. This research fits within a growing body of research that is exploring the utility of the instructional design theory of Realistic Mathematics Education (RME) for supporting the learning of undergraduate mathematics. For example, there are ongoing RME-guided instructional design projects at the undergraduate level in the areas of geometry (Zandieh & Rasmussen, 2010), linear algebra (Wawro, Rasmussen, Zandieh, Sweeney, & Larson, in press), and differential equations (Rasmussen, 2007). Like these projects, our research aims both to contribute to the ongoing development of RME and to contribute to specific knowledge about the learning and teaching of a particular mathematical topic (quotient groups).

As Weber and Larsen (2008) observe, it is well documented in the research literature that students find learning abstract algebra to be a difficult endeavor. More specifically, researchers have noted that students struggle with the quotient group concept (Dubinsky, Dautermann, Leron, & Zakis, 1994). Dubinsky et al., state that, "even if one begins with a very concrete group, the transition from the group to one of its quotients changes the nature of the elements (e.g., cosets) that are for her or him, undefined." (p. 268). In their study, they found that while a number of students were able to make progress in the case of the commutative group  $\mathbb{Z}_{18}$  (in terms of forming cosets and constructing an operation on these cosets) most of the students made much less progress in the context of the group of symmetries of an equilateral triangle. Further, Asiala, Dubinsky, Mathews, Morics, and Oktaç (1997) reported that while their second experimental course (featuring computer programming activities) seemed to support students in developing understanding of the primary components of the quotient group concept, only a small number of students (approximately one third) seemed to be able to put these components together.

On the other hand, Burn (1996) noted that the quotient group can be seen in the elementary example of even/odd parity, which suggests that in some sense the concept may be accessible to students at least in an informal sense. Dubinsky,

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Dautermann, Leron, and Zazkis (1994) counter by arguing that, “only someone who already possesses a good understanding of quotient groups can understand these examples as quotient groups” (p. 251). We agree with Dubinsky et al. (1997) that there is a significant difference between understanding parity and understanding quotient groups. However, we also take Burn’s (1996) observation as a sensible suggestion that parity can serve as an accessible starting point for developing the quotient group concept. The objective of the research reported here was to understand and support students in the difficult (Mason and Pimm, 1984) task of seeing the general quotient group concept in the specific example of parity. In particular, our goal was to develop a local instructional theory that would support students in reinventing this difficult concept beginning with their intuitive notions of parity.

## 1. Theoretical perspective

This research was influenced by the instructional design theory of Realistic Mathematics Education (RME) in two ways. First, the primary goal of the project is to develop an approach to the teaching of the quotient group concept that is consistent with the RME idea of *guided reinvention*. Gravemeijer and Doorman (1999) explain that the idea of guided reinvention “is to allow learners to come to regard the knowledge that they acquire as their own private knowledge, knowledge for which they themselves are responsible” (p. 116).

Second, our research and design work was influenced by two RME-related design heuristics. The first is the notion of an *emergent model*. According to Gravemeijer (1999) emergent models are used in RME to promote the evolution of formal knowledge from students’ informal knowledge. The idea is that the concept initially emerges as a *model of* students’ activity in experientially real problem situations<sup>1</sup> and then the concept later evolves into a *model for* more formal activity. The concept is considered a *model-of* when an expert observer can describe the students’ activity in terms of the concept. The concept is considered to be a *model-for* when students can use the concept to support their reasoning in a new situation. In this way, the transition from *model-of* to *model-for* can be seen as a transition to more general mathematical activity.

The second RME-related design heuristic that guided our work is the *proofs and refutations* framework proposed by Larsen and Zandieh (2007), based on Lakatos’ (1976) framework for mathematical discovery. Lakatos’ framework for explaining historical processes of mathematical discovery was adapted by Larsen and Zandieh to explain processes of guided reinvention in undergraduate mathematics education. Briefly, the process of proofs and refutations is one in which conjectures are revised (and concepts developed) as proofs are analyzed in light of counterexamples. Larsen and Zandieh argued that this framework could also support instructional design in that tasks could be designed to evoke these powerful kinds of mathematical activity. These two design heuristics (*emergent models* and *proofs and refutations*) were used to develop initial conjectures as to how the quotient group concept might be reinvented.

Our instructional design efforts and our analyses of the participating students’ mathematical activity were guided by Gravemeijer’s (1999) description of the ingredients of a local instructional theory. In particular, we worked to:

- Discover student strategies and ways of thinking that anticipate the formal concepts.
- Identify design principles for instructional activities that could be used to *evoke* these strategies and ways of thinking.
- Identify design principles for instructional activities could be used to *leverage* these strategies and ways of thinking to support the development of the formal concepts.

The local instructional theory (LIT) that we describe in this paper is the culmination of the various insights of these three types that have resulted from our analyses. This LIT describes a path by which students can reinvent the quotient group concept. This path is described in terms of the student strategies and ways of thinking that we identified as important milestones in the development of the fundamental ideas related to the quotient group concept.

## 2. Complexities of the quotient group concept

Before describing the preliminary instructional theory that supported our initial design experiment, it will be helpful to discuss some of the mathematical complexities of the quotient group concept. The elements of a quotient group are subsets of a given group. This collection of subsets must form a partition<sup>2</sup> of the original group. A number of mathematical issues emerge as one attempts to define an operation on this collection of subsets in a way that results in the formation of a group.

There are two reasonable ways to define an operation on the partition. In standard treatments, the product of the subsets  $A$  and  $B$  is taken to be the subset that contains the element  $ab$ , where  $a$  and  $b$  are elements of  $A$  and  $B$  respectively. Thus, the operation is performed by *multiplying representatives*. When defining the operation in terms of representatives, the most important question is whether the operation is well-defined (i.e., whether the answer is independent of the chosen representatives). An alternative way to multiply subsets is to define the product of two subsets,  $A$  and  $B$ , to be the set of *all products* of the form  $ab$ , where  $a$  is an element of  $A$  and  $b$  is an element of  $B$ . Subsequently, we refer to this operation as *set*

<sup>1</sup> Experientially real problem situations are not limited to situations one would encounter in everyday life. Instead the term refers to “that which at a certain stage common sense experiences as real” (Freudenthal, 1991, p. 17).

<sup>2</sup> A partition of a set is a collection of disjoint subsets, the union of which is the entire set.

*multiplication*. When the operation is defined in terms of set multiplication, there is no danger that the operation will not be well-defined. Instead, different questions emerge, like whether the resulting subset is a member of the partition.<sup>3</sup> The LIT that will be described here features set multiplication as the primary way of thinking of the operation on a partition of subsets of a group. The idea of multiplying representatives does emerge near the end of the reinvention process supported by the LIT, so it will be briefly considered when appropriate.

In order for a partition of a group to form a group under the operation of set multiplication, it is necessary for the partition to consist of cosets of a subgroup. A coset of a subgroup is formed by multiplying a group element by every element of the subgroup.<sup>4</sup> Such a partition of cosets does not always form a group under set multiplication. In order for a partition of cosets to form a group, it turns out to be necessary (and sufficient) for the subgroup to be a normal subgroup. If one multiplies cosets by multiplying representatives, normality ensures that the operation is well-defined. If one multiplies cosets using set multiplication, then normality ensures that the group axioms hold. There are a number of different but equivalent definitions of normality. The one that is anticipated by the LIT presented here characterizes normal subgroups as those whose left cosets and right cosets are equivalent as sets.<sup>5</sup>

The overarching goal of the LIT is to support students in developing the idea of partitioning a group into subsets that form a group and to discover necessary and sufficient conditions for this construction to be successful. The initial challenge is for students to develop the idea that a collection of subsets of a group could itself form a group under some operation. The challenge then shifts to determining how one needs to partition a group in order for this to work. A number of ideas have to be developed in order to address this challenge. Students must figure out that one of the subsets needs to be a subgroup, that the rest of the partition can then be only formed in one way (cosets), and that this subgroup must have a special property (normality). It is worth noting that this mathematical development of the quotient group concept reverses the standard treatment in which the cosets and normality are introduced (often with little motivation) *before* the quotient group concept. In the treatment presented here, the quotient group concept is the starting point, and the concepts of coset and normality emerge naturally as students figure out how to make the quotient group concept work.

### 3. Initial LIT for reinventing quotient groups

The design process began with a thought experiment in which we imagined a process by which students could reinvent the quotient group concept. This thought experiment was guided by our own understanding of the concept and by the *emergent models and proofs and refutations* design heuristics. It should be pointed out that our intent was to build on our previous work in which we developed an instructional theory for supporting the guided reinvention of the group and isomorphism concepts (see Larsen, 2009, in this issue). In our previous work, students began the reinvention process in the context of geometric symmetry, and we retained this context for the quotient group concept.

We began with the conjecture that the quotient group concept could emerge as a model of the students' informal mathematical activity as they searched for parity in the group  $D_8$  (the symmetries of a square). This conjecture was inspired by the fact that in our earlier research, students often noticed (in the context of the symmetries of a triangle) a specific pattern in how flips and rotations interact (two rotations is always a rotation, two flips is always a rotation, and a combination of a flip and a rotation is always a flip). This is the same structure that can be seen with even and odd integers. Thus, we anticipated that students would partition the group into flips and rotations as a natural analog to partitioning the integers into evens and odds. We expected that the students could then further mathematize this activity first by considering whether the even/odd partition they created formed a group in some sense, and then later by generalizing to more complex partitions (e.g., partitions into four subsets consisting of pairs of elements). The group of symmetries of a square is well suited to support the investigation of these more complex partitions because it has four two-element subgroups that are not normal. Thus, it admits a number of sensible partitions that do not form quotient groups.

Recall that the transition from a model-of to a model-for can be associated with a transition from referential to general activity. We anticipated that the students' mathematical activity would become more general in two ways. First, the students' activity would generalize from initial work focused on parity to a more general focus on groups formed via partitioning. Second, while we expected the students' initial partitioning activity to be closely tied to the context of the symmetries of a square, the intent was for them to regularly consider the generalizability of their activity. Specifically, they were encouraged to explore the necessity of any properties they observed in this starting point context. Our initial local instructional theory drew on the proofs and refutations heuristic (Larsen and Zandieh, 2007) to support this second kind of advancement of the students' mathematical activity. As the students analyzed various partitions we expected that, by considering examples and non-examples, the students might develop conjectures as to what conditions are needed to make a partition function as a group. As students attempted to verify that certain partitions formed groups, we expected that, by analyzing this proving activity, they could gain insight into what conditions were necessary to make it work. It should be noted that we were quite unsure how the notion of normality would emerge from the students' mathematical activity. Therefore, it was important

<sup>3</sup> For example, if one partitions a group into two-element subsets, it is possible for a product of two subsets to generate a subset with four elements.

<sup>4</sup> If  $H$  is a subgroup of the group  $G$  and  $g$  is an element of  $G$ , then  $gH = \{gh : h \in H\}$  is a coset of  $H$ .

<sup>5</sup> Formally, a subgroup  $H$  of  $G$  is said to be *normal* in  $G$  if for every element  $g$  in  $G$ ,  $gH = Hg$ .

for us (as we began our first design experiment) to monitor the participating students' activity closely in order to identify informal strategies or ways of thinking that anticipated the idea of normality.

#### 4. Research method

The research reported here consists of a series of design experiments (The Design-Based Research Collective, 2003). Analyses of data collected during each experiment informed the ongoing development of the emerging local instructional theory and the design of specific instructional tasks. Here we focus on the first three phases of the research and design cycle:

*Phase 1: Initial Design Experiment.* The first phase consisted of a design experiment conducted with a pair of undergraduate students. Rick was an undergraduate mathematics major while Sara was an in-service middle school teacher. The students completed their participation in the design experiment *just before* enrolling in their first abstract algebra course. Both students were quite strong mathematically (both would go on to earn an A in their abstract algebra course). These students were selected because they were known to be articulate, creative, and thoughtful – and thus were expected to be a good source of ideas regarding how students might navigate the various complexities of the quotient group concept. The teacher/researcher met with the two students for eleven sessions, each lasting approximately 90 min. Each session was videotaped by one of the two graduate students who also participated in debriefing and planning meetings after each session. Additionally, all of the students' written work was collected.

*Phase 2: Experimental Teaching.* The instructional theory and initial instructional sequence that resulted from the first design experiment was adapted and implemented in a regular group theory class by the first author. Methodologically, the best descriptor of this phase is *experimental teaching* (Steffe & Thompson, 2000). This phase did not include data collection, but did result in an initial set of lesson plans and instructional notes for use in the subsequent whole class teaching experiment.

*Phase 3: Whole Class Teaching Experiment.* The third phase was a whole class teaching experiment in a regular group theory course in collaboration with a research mathematician (Cobb, 2000). Data was collected during the seventh and eighth week of a ten-week term.<sup>6</sup> Three sessions (those focused on the quotient group concept) of the class were video-recorded using two cameras (filming two different groups during small group work). Planning and debriefing meetings with the instructor were held before and after each of these class sessions (and weekly during the rest of the term). These meetings were also video-recorded.

Data analysis consisted of both ongoing and retrospective analyses. Ongoing analyses were conducted between instructional sessions to identify student strategies and ways of thinking that anticipated more formal quotient group related ideas (e.g., cosets, normality) and generate conjectures as to how these ways of thinking could be leveraged to develop the formal ideas. These conjectures were then tested during subsequent instructional sessions. For example, a student was able to leverage the identity property (of the subgroup used to construct a quotient group) to create an algorithm for partitioning a group. This algorithm strongly anticipated the definition of coset.

The retrospective analysis involved multiple cycles of video analysis (Cobb & Whitenack, 1996; Lesh & Lehrer, 2000). The first phase of each retrospective analysis was focused on identifying student strategies and ways of thinking that anticipated the quotient group concept. The second phase of analysis was focused on identifying principles for evoking these strategies and ways of thinking by attempting to understand what about the task design or the discourse (among the students or between the students and the teacher/researcher) could explain the emergence of the strategy or way of thinking. Essentially, the goal of this phase of analysis was to construct explanations for the emergence of the students' most productive and powerful ideas. The third phase of analysis was similar, but it focused on identifying principles for leveraging the students' informal ideas in order to promote the development of the more powerful formal concepts. The goal was to understand what features of the task design or discourse supported students in building on their ideas and moving toward the formal concepts. For example, we noticed that students in the initial design experiment were able to gain considerable traction at key points in the reinvention process by focusing on the identity property of the subgroup used to partition the group. This idea was leveraged heavily as we revised tasks and modified them through experimentation in the regular group theory course.

The result of these analyses is a refined LIT and a sequence of instructional tasks that is consistent with this LIT. The current version of this instructional theory will be presented in the following sections in the form of a sequence of key steps in the process of reinventing the quotient group concept. This sequence is described using theoretical constructs from RME and is illustrated with examples of students' and teachers' mathematical activity (see Johnson, in this issue) drawn from data collected during both the initial design experiment and the whole-class teaching experiment.

#### 5. Local instructional theory

##### Step 0: Starting point context

In all phases of the research process, the students began the quotient group unit having reinvented the group concept through an RME-inspired inquiry approach (see Larsen, in this issue). In the initial design experiment with a pair of students,

<sup>6</sup> For the first six weeks, the instructor implemented an RME-inspired inquiry oriented curriculum that was the result of some of our earlier design work (Larsen, Johnson, & Bartlo, in press).

<b>+</b>	<b>EVEN</b>	<b>ODD</b>
<b>EVEN</b>	<b>EVEN</b>	<b>ODD</b>
<b>ODD</b>	<b>ODD</b>	<b>EVEN</b>

Fig. 1. Even/odd operation table.

this process took place over the course of the first five sessions and culminated with the formulation of a definition of group. The students then used this definition to evaluate several set/operation pairs to determine whether they were groups. Two such pairs were particularly relevant to the subsequent process of reinventing the quotient group concept: the set of integers under addition, and the Even/Odd group shown in Fig. 1. From an expert perspective, the group shown in Fig. 1 is a quotient group. However, the fact that the students were able to verify that this table represents a group should not be interpreted to mean that they were making sense of it in a way consistent with the quotient group concept. In fact, in subsequent sections we present evidence that the students likely initially saw this table as merely an efficient representation of the full group  $(\mathbb{Z}, +)$ .

After the students had practiced determining whether various set/operation pairs formed groups, they were asked whether any of the subsets of the symmetries of a square were themselves groups (and the term *subgroup* was introduced). The idea was that this experience would not only provide more practice reasoning with their new definition of group, but would also be helpful when they began working to partition  $D_8$  into groups of subsets. In the initial design experiment, the students used additive notation<sup>7</sup> and their set of symbols for  $D_8$  was  $\{0, R, 2R, 3R, F, F+R, F+2R, F+3R\}$  where  $R$  represented a  $90^\circ$  clockwise rotation and  $F$  a flip across the vertical axis of symmetry. Interestingly, Rick constructed one of these subgroups,  $\{0, F, 2R, F+2R\}$  by collecting the elements that he saw as “even” in the sense that they involved an even multiple of  $R$ . Rick and Sara would go on to identify all of the subgroups of  $D_8$ ,  $D_6$ , and the integers (under addition).

#### Step 1: Identifying evens and odds in $D_8$

As stated in the description of the conjectured LIT, the approach taken to reinventing the quotient group concept is to generalize and abstract the notion of parity. With this in mind, the first step of the reinvention process is for students to examine a finite group that they are intimately familiar with in order to identify analogs to the parity observed in the integers. Because the LIT was intended to extend our LIT for the group concept (Larsen, in this issue), which takes geometric symmetry as its point of departure, the group of symmetries of a square was selected as the starting point.<sup>8</sup> In the initial design experiment, the task used to launch the reinvention of the quotient group concept was presented in the form of the question, “Is there anything like evens and odds in that table?” (where the ‘table’ refers to the table shown in Fig. 2).

There is more than one way to conceptualize this task, because there are several aspects of parity on which one can focus. Our intention was that students would focus on the way that evens and odds interact under addition ( $E+E=E$ ,  $O+O=E$ ,  $O+E=O$ ,  $E+O=O$ ). However, we found that students were just as likely to focus on evenness or oddness as characteristics of individual elements. For example, Rick immediately identified the evens as  $\{0, F, 2R, F+2R\}$ , which is not surprising given that he initially discovered this subgroup by collecting those symbols containing even multiples of  $R$ . Rick explained that “the way I talk about evens and odds is evens are two times something.” Given Rick’s method for constructing the subgroup  $\{0, F, 2R, F+2R\}$  we anticipated that he might identify this as the set of evens, and responded by asking if there was another way to see evens and odds in the table. Sara answered by describing how she was thinking about evens and odds, saying, “Well an even plus an even is an even. . . an odd plus an odd is an even” and “an odd plus an even is always an odd.” Rick then emphasized yet another aspect of parity (the alternating pattern on the number line) when he said, “the way we organized it (listing the elements on the table in the order  $0, R, 2R, 3R, F, F+R, F+2R, F+3R$ ) they go back and forth just like we think of evens going back and forth.”

We can understand these different ways of conceptualizing this task in terms of metaphorical thinking (Black, 1977). These different perspectives can be distinguished in terms of the aspects of even/odd parity that are highlighted and the aspects that are suppressed. Rick’s first statement emphasizes evenness as a characteristic that relates to the structure of individual numbers. Sara’s statement, about how evens and odds interact under addition, emphasizes seeing parity as a

<sup>7</sup> In Phase 2 and Phase 3 (the whole-class settings) the students used multiplicative notation.

<sup>8</sup> This group was selected because it contains both normal and non-normal two-element subgroups. As a result, it provides a good context for identifying normality as a necessary condition for quotient group construction.



	O	R	2R	3R	F	F+R	F+2R	F+3R
O	O	R	2R	3R	F	F+R	F+2R	F+3R
R	R	2R	3R	O	<del>F+R</del>	F	F+R	F+2R
2R	2R	3R	O	R	F+2R	F+3R	F	F+R
3R	3R	O	R	2R	F+R	F+2R	F+3R	F
F	F	F+R	F+2R	F+3R	O	R	2R	3R
F+R	F+R	F+2R	F+3R	F	R	2R	3R	O
F+2R	F+2R	F+3R	F	F+R	2R	3R	O	R
F+3R	F+3R	F	F+R	F+2R	3R	O	R	2R

$R + F + 3R$   
 $F + 6R = F + 2R$   
 $O \ 2R = O \ 3R$

$R + F + R$   
 $F + 3R + R = F$   
 $R + F = F + 3R$

$R + F + R + R$   
 $F + 3R + R + R$   
 $F + R$

Fig. 2. Operation table for  $D_8$ .

characteristic held by members of the two subsets under an operation. Finally, Rick's comment about "going back and forth" emphasizes the alternating pattern that is observed in evens and odds on the number line. Any of these ways of thinking may lead to the partitioning of the group ( $D_8$  in this case) into two sets, one of which is a subgroup. We found that Sara's perspective is the one that ultimately provides the most leverage in terms of generalizing from parity to partitions of subsets that form groups under set multiplication. However, we have observed students successfully using these other perspectives to guide them in creating partitions to test.

In terms of the instructional design heuristics that guided our work, this first step represents the initial emergence of the quotient group concept as a model of the students' mathematical activity. From an expert observer's perspective, the students' activity can be seen as consistent with the quotient group concept in that they are partitioning a group into subsets (cosets) that form a group. However, at this point there are many aspects of the quotient group concept that are implicit or absent in the students' activity, including the key idea that this set of subsets forms a group under some operation.

### Step 2: Viewing even/odd partitions as groups of subsets

Our initial instructional theory called for a shift in the students' ways of thinking about their even/odd partitions – away from the specific idea of parity and toward the more general notion of group. The idea was to first support students in realizing that these partitions did form groups made up of subsets, and then to get them to generalize this idea by asking them to construct more complex groups of subsets. This notion, that the even/odd partitions in fact formed groups whose elements were subsets, was not at all obvious to the participating students. In the following excerpt we see Sara and Rick coming to terms with this idea. During this interaction the students had two operation tables in front of them: one was the full table for  $D_8$  and the other was a  $2 \times 2$  table representing the partition of  $D_8$  into flips and rotations. Note that it was a struggle for the students to generate the  $2 \times 2$  table even when asked to produce something analogous to the Even/Odd table in Fig. 1 above (Fig. 3).

Teacher/Researcher: So this one is a group, right? [Points to the full  $D_8$  table]

Sara: Yes.

Teacher/Researcher: So what are the elements of this group?

Rick: [Points to each of the column headings on the full  $D_8$  table]

Teacher/Researcher: The things along the side of the, the headings, right?

Rick: Yeah, those eight guys. Our eight moves.

Teacher/Researcher: So what would the elements be of this one? [Points to the  $2 \times 2$  table] What would the elements of this group be?

Rick: [Again points to each of the column headings on the full  $D_8$  table]

The students explained that in their view the  $2 \times 2$  table was just an abbreviated way to represent the full operation table. So even though they were seeing parity in terms of how these two subsets interacted, they did not think of it as a group made up of subsets as they had (apparently) done in the case of the even and odd integers. As a way to challenge the students to think in terms of a group made up of subsets, the teacher/researcher asked them how many elements there were in the group represented by the  $2 \times 2$  table and asked them to consider the possibility that there were only two.

Diagram illustrating the XOR operation using a 2x2 matrix:

	R	F
R	R	F
F	F	R

Below the matrix, the XOR operation is defined as:

$$R \oplus F = (R \wedge \neg F) \vee (\neg R \wedge F)$$

**Fig. 3.** A  $2 \times 2$  flip/rotation table and the full 8-element operation table for  $D_8$ .

*Teacher/Researcher:* How many elements does this group have in it? [Points to the full D8 table]

**Rick:** Eight.

*Teacher/Researcher:* How many does this one have [points to the  $2 \times 2$  table]?

**Rick:** I don't think this is a group. I don't think.

*Sara:* It does have eight.

*Teacher/Researcher:* What if I said it had two?

**Rick:** Then it wouldn't be a group.

*Sara:* Well if you want to make meta-groups.

*Teacher/Researcher:* Ok, let's make meta-groups.

**Rick:** No wait, maybe it would be a group. It would be a group for me.

*Sara:* Yeah.

*Teacher/Researcher:* What would the elements of the group be?

**Rick:** Rotations and flips. My identity would be rotations.

This was an important moment in the students' reinvention of the quotient group concept. It was at this point that they began to think in terms of constructing groups whose elements were subsets. This is an example of vertical mathematization (Rasmussen, Zandieh, King, & Teppo, 2005), because the students had moved from merely organizing their set of symmetries (and symbols) to creating a new mathematical reality in which subsets could be considered to be elements that could be operated on to form a group structure.

Recall that a primary goal of our analyses was to uncover student strategies or ways of thinking that anticipated the formal quotient group concept and to discover principles for evoking these ways of thinking. The question about how many elements there were in the group represented by the  $2 \times 2$  table seemed to be a high-leverage question in terms of supporting the students in shifting to thinking in terms of a group whose elements were subsets. When asked to consider whether it could be thought of as a group consisting of two elements, the students were able to describe a way in which this could make sense. Sara *introduced* the suggestive (and non-standard) term “meta group,” and then the students went on to explain that the two elements of the group were the flips and the rotations, where the set of rotations was the identity element. When asked what kind of objects the elements of the “meta group” were, the students responded that they were “types” or “classes” of moves. Until the teacher/researcher introduced the term “quotient group” (near the end of the study), the student-created term “meta group” was used to refer to the groups of subsets that Rick and Sara constructed by partitioning groups.

When the students had made sense of the idea that the partitions could form two-element groups of subsets, the teacher/researcher introduced colored index cards as a way to support the students in thinking about the subsets as elements. He started by forming a  $2 \times 2$  table using four blue index cards and four yellow index cards. The set of rotations,  $\{0, R, 2R, 3R\}$ , was listed on each yellow card and the set of flips,  $\{F, F+R, F+2R, F+3R\}$  was listed on each blue card. The teacher then arranged these cards in the form of an operation table.

The students then formed similar tables (using colored cards) for the other two even/odd partitions<sup>9</sup> they had previously identified and verified that they formed groups. This activity of *proving* that the partitions formed groups of subsets was crucial in terms of promoting a shift away from thinking of these partitions only in terms of parity. In the whole-class teaching experiment, the instructor (Dr. James) led the class through such a process (for each of the three valid partitions) during a whole-class discussion. The following transcript excerpt and Fig. 4 represent the first of these collective proving

<sup>9</sup> In the initial design experiment and in the whole class teaching experiment, the students partitioned the group into evens and odds in three ways, one for each of the three four-element subgroups of  $D_8$  (the subgroup was always considered the set of evens).



Fig. 4. Verifying an even/odd partition of  $D_8$  forms a two-element group.

processes. (As in the initial design experiment, Dr. James introduced colored index cards in order to help students view the subsets as elements while still keeping track of the group elements included in each subset.)

Dr. James: So let's start with the one, I think that every single group found this one first. The role of the evens is being played by the rotations. Okay, so this has  $I$ ,  $R$ ,  $R^2$ , and  $R^3$  on it. That's the evens. And the role of the odds is played by everyone else. Which is what? The flips. So we have  $F$ ,  $FR$ ,  $FR^2$ , and  $FR^3$  on this card. And then when you play that out you get this sort of a table. . . All right, let's talk through this table and discuss whether or not this is in fact a group. Does this structure describe a group? So what would we need to check to convince ourselves, or convince a skeptic, that this is in fact describing a group?

Dan: Closure.

Dr. James: Closure, okay. So first of all do we have closure here, and if so how can you tell by looking at that?

Kate: There's no new color.

Dr. James: There's no new color. So you've got the two colors, there's no new color on the table so therefore there is closure. I'll buy that. What else do we have to check to make sure that this is really a group?

Phil: Identity.

Dr. James: Identity, okay. What's that gonna –

Steve: That lovely lavender stripe that runs on the diagonal. That's going to be our identity.

Dr. James: I'm color blind, so would this be lavender? Is that right? So you're saying this one acts like the identity, talk me through that. Like, what am I looking for to make sure that acts like the identity?

Phil: The element times the identity is the element.

Dr. James: So, an element times the identity should be that element. So the entries in each row should match the column heading or something like that. So do you guys buy that, if you stare at this that lavender looks like the identity element? In every case that needs to work? How about, um, what else do we need to check?

Students [Simultaneously]: Inverses. Associativity.

Dr. James: Well, let's go with inverses first. What am I looking for with inverses? How can I look at the table and see whether or not inverses exist?

Phil: An element times its inverse is the identity.

Dr. James: Ok, so any element times itself, in this table. Any element times itself, that's the diagonal entries, is lavender, which we've previously decided lavender was the identity. Great! So every row and every column has the identity in it so that means I guess every element has an inverse. And what's the final thing to be checked?

Kate: Associativity.

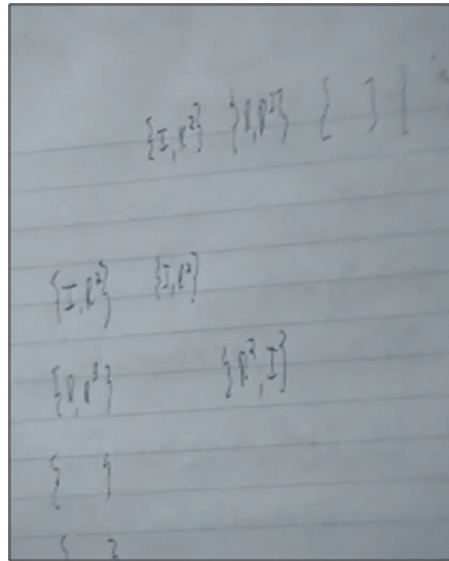
Dr. James: Associativity. So think about that one. Ok good, so it's associative. Does anyone want to take a stab at why associativity would hold? Did anyone actually check all eight cases?

This second step is an important one in terms of the model-of/model-for transition because the students' activity (verifying that their even/odd partitions are groups) explicitly focuses on the notion of a group whose elements are subsets. One way to describe this step is that the students are mathematizing their previous activity (forming even/odd partitions) by analyzing their partitions using the group concept. Note that in this sense, the *group concept* is functioning as *model for* supporting the students' partitioning activity. However, because the students' goal when they partitioned the group was not to create a group of subsets, it is only from an outside expert's perspective that their activity can be seen as exemplifying the quotient group concept. In fact, it is unclear how exactly the students in the previous excerpt are thinking about the operation represented by the table that they verified represented a group. At this point, it is quite possible that some students are thinking of the operation in terms of multiplying representatives (e.g. a flip combined with a rotation is a flip) while others are thinking in terms of set multiplication.

### Step 3 Partitioning $D_8$ to construct a four-element "meta group"

The next step in the instructional theory supports the model-of/model-for transition in that the students explicitly take on the task of forming a *group* made up of subsets of  $D_8$ , thus generalizing the notion of even/odd parity. Previously, the students' partitioning activity was focused on parity rather than on the group concept. This new phase of the reinvention process is launched by asking the students if they can partition  $D_8$  to form a group containing more than two subsets. For





**Fig. 5.** Constructing a group comprised of four two-element subsets.

example, in the whole-class teaching experiment, Dr. James observed that students had asked (during the previous step) whether they had to break the group into two subsets and used this question to motivate this next step.

Dr. James: Some groups had a further question, which was, “do you have to just split it into two sets?” Could you maybe make a group by splitting into other smaller subsets or bigger subsets or just different than four and four? . . . Okay, for the sake of clarity let me emphasize something. We are not now trying to do even and odds any more. We are looking for subsets that just make a group. . . . Okay, just one more time, in the interest of really pushing you in the right direction, we’re actually looking for a solution that has four subsets. See if you can find four pairs of elements that form a group.

In the following excerpt, three students are in the process of successfully constructing the group consisting of the subsets  $\{I, R^2\}$ ,  $\{R, R^3\}$ ,  $\{F, FR^2\}$ , and  $\{FR, FR^3\}$ . Meanwhile, another student in the group is struggling to move beyond the idea of partitioning the group into evens and odds. He specifically asks what the criteria are for this new partition, and another student explains the new “kinda weird” objective.

Kevin: So you are trying to say you have  $I, R^2$  and you want to find another pair, is that right? Maybe this one is even and you find an odd.

Charles: But we need four [sic] more pairs. We need a pair here, a pair here, a pair here. And if you put  $R, R^3$  here,  $R, R^3$  here, you’re gonna get, this could be  $R^2$ . Or it’s just  $R^2$ . Or it could be  $I$ . Oh that works though, I mean, I guess. Right? . . . You’re gonna have that somewhere. So here you’re gonna have  $R, R^3$ . So I guess that’s okay that it equals  $R^2$ . So then, what about the flips? We do  $F$  and  $FR^2$  as one. And  $FR, FR^3$ . So that makes sense. I think that’s gotta be.

Lee:  $F, FR$  squared?

Charles: Yeah.

Kevin: What kind of property does this satisfy? Does this satisfy the even or odd?

Charles: No. Really it’s more complicated than even and odd. That’s the problem, they’re trying to get us away from even and odd.

Kevin: But what’s the requirement to do this? You try to find subgroups?

Charles: You just try to find subsets that make kind of a, that map onto another one of the subsets. It’s kind of a group of subsets, sort of. It’s kinda weird.

Charles tested his partition as it was being constructed by building an operation table (Fig. 5).

For example, Charles multiplied the set  $\{R, R^3\}$  by the set  $\{R, R^3\}$  by multiplying each element in the first subset by each element in the second subset to obtain the subset  $\{I, R^2\}$ . At this point, we can see clearly that he was thinking in terms of set multiplication rather than multiplying representatives. Amber, the undergraduate TA for the course, dropped in on this small group of students and asked them about the partition that they had found.

Charles: We started with  $I, R^2$  because that seemed to be a good starting place.

Amber: Out of curiosity, why did you think it was a good starting place? What was it that made you guys want to choose that?

Charles: We know that it mapped onto itself. Like it was kind of an identity of a sort. So  $I$  combined with  $I$  would equal  $I$ .  $R^2$  with  $I$ . So basically all the combinations. . .

Amber: And  $R^2$  with  $R^2$ ?

Charles: It maps onto  $I$ . So then it made sense to us to group the other  $R$ ’s. . . So we grouped them together to see if we combined those two what we would get. And it turned out to be  $I, R^2$ . That was our first subset.

Amber: So when you did the  $R$  and the  $R^3$  and the  $F$  and the  $FR^2$  and the  $FR$  and the  $FR^2$ , you guys did that arbitrarily is what you’re saying?

Jade: No.

Amber: No, okay. Okay, so how did you decide, if it wasn’t arbitrary?

Jade: So when I look at this table you can divide this small square. . .

Amber: Okay so you’re taking  $I, R^2, FR$  and  $FR^3$  and you’re dividing that in half? . . . So you’re kinda breaking apart one that we already had. And that seemed to be working? . . . Okay, and that’s why you chose them? Okay. Okay, great.

Charles: So it wasn’t completely arbitrary. . . . The first step was finding an identity set. And the second step was breaking up the rotations because we knew the rotations were kind of separate from the flips. And there’s kinda, I don’t know, natural counterparts in the flips, too.

In this excerpt, the students explain that they first selected the subgroup  $\{I, R^2\}$  because it mapped onto itself ( $\{I, R^2\}\{I, R^2\} = \{I, R^2\}$ ) and it thus behaved like an identity element. Then they explain that they grouped the other two rotations  $\{R, R^3\}$  and found that the product of this subset with itself was the subgroup  $\{I, R^2\}$ . Finally they explain that they selected the subset  $\{FR, FR^3\}$  by partitioning the quadrant of their operation table for  $D_8$  that contained the operation table for the subgroup  $\{I, R^2, FR, FR^3\}$ . Note that this table for  $D_8$  was divided into quadrants as a result of the earlier task in which the students partitioned  $D_8$  into evens and odds. From this discussion we can see that students can leverage some combination of geometric intuition and their earlier activity to guide them in constructing a partition of  $D_8$  into four subsets that form a group.

This third step of the reinvention process presents two challenges for the students. The first challenge is to shift away from looking for parity in  $D_8$  toward explicitly and intentionally forming groups made up of subsets. This shift to intentionally constructing a group comprised of subsets is particularly important because, at this point, one can accurately model the students' activity with the quotient group concept. This sets the stage for the hard work of establishing the mathematical rules that govern such activity, work that is necessary to support the transition of the quotient group concept to a model for more formal activity.

The second challenge faced by the students is in coming up with a partition that works. Of the many ways to partition  $D_8$  into four subsets each containing two elements, only one of them forms a group. The search for such a partition both supports, and is supported, by activity consistent with the process of proofs and refutations. As students attempt to partition the group they test each partition by attempting to prove that it works (by forming an operation table). By analyzing these proof attempts, the students make and test conjectures about what is necessary to make the process work. This third step of the reinvention process typically results in the students successfully forming the one possible quotient group with four elements, and producing some conjectures about necessary conditions. The following steps focus on establishing necessary conditions.

#### Step 4 (A): Identifying necessary conditions part 1 – Identity subset must be a subgroup

We found in the design experiments that the students very quickly conjectured that one of the subsets must be a subgroup in order for a partition to form a group (or be analogous to the parity in the integers). Indeed students will often use this fact to guide them as they attempt to construct partitions. For example, in the following excerpt from the whole class teaching experiment, a student explains that there can be no more than three ways to partition  $D_8$  into evens and odds because  $D_8$  has only three subgroups of order four. Here the conjecture that the identity subset must be a subgroup emerged early in the reinvention process as a result of Dr. James' spontaneous question about whether the students felt that they had found all of the possible even/odd partitions.

Dr. James: Does anyone have a feeling that there may be more or not more?

Jenna: There's not going to be any more because you have to have one be a subgroup. And everything else will fall out. Everything that's not in that subgroup.

Dr. James: Okay. This is an interesting point being raised here. So the claim is that one of the two sets actually has to be a subgroup. Can you speak to that more? Why do you think that one of them needs to be a subgroup?

Jenna: If it's a subgroup, then it with itself is just going to make that subgroup back. And then you're going to have everything left for the rest of it.

Dr. James: And you're looking for one element that with itself, the evens. The evens with itself is the evens.

Jenna: And then everything will, everything else on that line [the second subset] I guess, will be everything else.

Dr. James: Okay, so you claim that based on that reason, that some element with itself has to equal itself. You're making the case that you need to have a subgroup to be one of your sets. And everything that's not in that subgroup will be in the other set.

Jenna: Yeah, so that subgroup will act as the identity.

The idea that one of the subsets must act as an identity subset is crucial throughout the reinvention process. The task of proving that the identity subset must be a subgroup has at least two benefits in terms of supporting the reinvention of the quotient group concept. First, it establishes with certainty that the students need only consider partitions in which one of the subsets is a subgroup, setting the stage for establishing coset formation as the only viable way to partition a group into a group of subsets. Second, the activity of proving that the identity subset must satisfy the closure, identity, and inverse properties involves explicitly leveraging the identity property of this subset. This is a strategy that later supports the students' activity related to coset formation and the normality condition.

In the excerpt above, Jenna alludes to the necessity of closure for the product of the identity subset with itself to be the identity subset. Students can make similar arguments to establish that it is necessary for the identity subset to contain the identity of the original group, and that it is necessary for the identity subset to be closed under inverses. In each case, students typically proceed by way of contradiction. For example, in the following excerpt from a whole-class discussion, a student explains how assuming that the inverse of some element of the identity subset is actually in a different subset will lead to a contradiction. As the student provides the argument, Dr. James explains and elaborates at the board, drawing the partial operation table shown in Fig. 6.

Faith: Well if you started out and you have that first one be equal to like  $\{I, A\}$ .

Dr. James: Ok, sure. So back up, we do know that  $I$  is in the identity subset.

Faith: And then if you originally say that  $A$  times  $B$  will be the identity. Well not times,  $AB$  will be the identity.



Fig. 6. Proving that the identity subset must be closed under inverses.

Dr. James: Ok, so let me just interrupt a little bit. So we're trying to convince ourselves say that  $A^{-1}$  also has to be in this identity set. So were pretending that  $B$  is the inverse of  $A$  and putting  $B$ , we want to make sure the inverse of  $A$  is in here, but pretend for the moment that its not in there, it's over here. All right, lay it out for us.

Faith: Then when in that second column you're gonna end up with  $AB$  and that will put  $I$  in there.

Dr. James: So if the inverse of  $A$  ends up over here, then when you work out what belongs in here, you will at some point being multiplying  $A$  by  $A^{-1}$  giving you the identity in here. Now, what's wrong with that?

Student: You have the identity in the first one.

Dr. James: The identity's over here. So this is already the identity subset. This is some other subset by the Sudoku property. So, what's  $I$  doing here?

In her argument, Faith<sup>10</sup> leverages the fact that the product of the identity subset with itself must be the identity subset. Then, using the previously proven fact that the identity subset must contain the identity element, she is able to deduce that the identity element of the original group must be in the set that appears in the first row of the operation table as the product of the identity subset with itself (see Fig. 6). Then using the assumption that the inverse of  $A$  is actually in another subset, she is able to deduce that the identity element of the original group must also be contained in a second subset that appears in the first row of the table (see Fig. 6). This leads to a contradiction, because all of the subsets in any row are supposed to be different (by the Sudoku property of group operation tables). This strategy of leveraging the identity property of the identity subset to deduce necessary conditions is a recurring theme in the LIT. In general the approach is to assume that a partition forms a quotient group, and then leverage the fact that one of the subsets must act as an identity element to deduce conditions that are necessary for a partition to be successful.

At the conclusion of this discussion, Dr. James asked about the generality of this argument: "By the way, in this example we had two elements in this set. Do we need to have a two element set?"<sup>11</sup> This is a consistent theme in the LIT. Students reason initially in the specific context of  $D_8$  and then consider whether their reasoning generalizes. The sample task shown in Fig. 7 also asks students to consider whether their reasoning makes sense in the general case.

#### Step 4 (B): Identifying necessary conditions part 2 – Coset formation

After it has been established that one of the subsets must be a subgroup, the next step is to determine how to partition the rest of the group. In the following excerpt from the initial design experiment, the teacher/researcher asks Sara and Rick whether the remaining subsets are predetermined after the subgroup is selected.

Teacher/researcher: So here's a question. So suppose I give you some group and I ask you to build a meta-group. And you chose a subgroup, which you've kind of been doing it that way right? The first thing you've done is picked a subgroup and then done the rest from there.

Sara: Sorta like the seed.

Sean: So... We're going to do a  $\{0, 2R\}$  and then we'll go from there. So here's a question. Once you pick your subgroup, is everything predetermined? Or are there still options? In terms of what the other elements of the meta-group are?

Sara: I'm going to conjecture that it's predetermined.

Rick went on to develop a method for calculating the second element of any subset when given the first element. Each time he enacted this process he produced an inscription like that shown in Fig. 8. These inscriptions and the process itself (described in the following excerpt) strongly anticipate the formal notion of a coset.

Teacher/Researcher: So can you explain this process that you just did?

Rick: I take my identity, right? Zero and  $2R$ . Then you give me any other element in the entire group. Name one.

Teacher/Researcher:  $F + 3R$ .

Rick:  $F + 3R$ . I don't know this element [the second element paired with  $F + 3R$ ]. But I know that  $F + 3R$ , this subset, if I add it back through this subset [the identity subset] I've got to get this one back [the subset containing  $F + 3R$ ]. So I take  $F + 3R$ , add zero. I get  $F + 3R$ . I'm happy. I take  $F + 3R$ , I add  $2R$ , I get  $F + R$ . I'm like, oh I see,  $F + R$  goes back here in my little subset. And then I can check it by adding these [adding  $F + R$  to Zero and  $2R$ ]. And they give me these answers again. This one gives me these two answers. And I know I'm right.

<sup>10</sup> The Sudoku property refers to the fact that in the operation table for a group each element must appear exactly once in each row and column.

<sup>11</sup> Recall that, in fact, the students had already worked with examples in which the subsets contained four elements (the even/odd partitions).

### Does the identity element of a quotient group need to contain the inverse of each of its elements? Why?

Think about this for the case of  $D_8$  and then see if your reasoning makes sense for any group.

Hint: Suppose that  $g$  is in the identity (blue) subset but its inverse is in the yellow subset

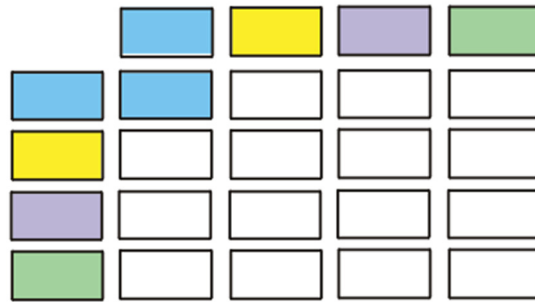


Fig. 7. A handout to support proving that the identity subset must be a subgroup.

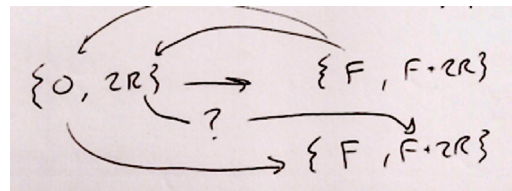


Fig. 8. Rick's algorithm for partitioning a group (forming the coset  $F + \{0, 2R\}$ ).

In the example shown in Fig. 8, Rick started with the subgroup  $\{0, 2R\}$  of  $D_8$ . He then chose the element  $F$  from outside the subgroup. He stated that the result of combining the subset  $\{F, \cdot\}$  with the identity subset  $\{0, 2R\}$  must be the subset  $\{F, \cdot\}$ . Then adding  $F$  to  $0$  and  $F$  to  $2R$ , he obtained the subset  $\{F, F+2R\}$  and then deduced that  $F$  must be paired with  $F+2R$ .

This is exactly the kind of informal strategy this initial design experiment was intended to discover. This strategy clearly anticipates the formal definition of coset. (Notice that the inscription shown in Fig. 8 illustrates the idea that the subset is the result of adding an element to each element of the subgroup – the coset formation process.<sup>12</sup>) Further, it appears that this strategy can be evoked by a task design (see the sample task in Fig. 9) that emphasizes a focus on the identity property of the subgroup.

The process Rick developed to partition a group after selecting a subgroup did not require anything as complicated as the full proofs and refutations process described by Larsen and Zandieh (2007), but it does similarly illustrate the role that proving can play in defining. Essentially what Rick did was execute part of the verification process that would be required to prove that a given partition formed a group. One would have to verify that the subgroup acted as an identity element with respect to each subset of the partition. Rick worked through his verification process for the subset he was attempting to construct by using an unknown (blank space) to stand in for the second element, and then carrying out the set multiplication.<sup>13</sup> He then used his awareness of what the result of this product would have to be (given that the subgroup was acting as an identity element) to determine the second element of the subset.

After Rick introduced his process for creating a successful partition, the teacher/researcher introduced the term “coset” to describe the subsets created using this process. Similarly, in the whole-class teaching experiment, the instructor introduced this terminology immediately after the students developed their method for partitioning the group.

#### Step 4 (C): Identifying necessary conditions part 3 – Normality

In the initial design experiment, the students successfully constructed a four-element quotient group and experimented with several other partitions that did not work. They were asked to construct a partial operation table (using colored index

<sup>12</sup> Rick's algorithm is consistent with the definition of a left coset.

<sup>13</sup> Given that Rick and Sara were using additive notation, set multiplication here really means set addition.

### How do you figure out what the other subsets need to be?

a) Suppose you want to use the subgroup  $\{I, FR\}$ . Figure out which element would have to be paired with  $R$ .

	$\{I, FR\}$			
$\{I, FR\}$	$\{I, FR\}$			
$\{R ?\}$				

b) Suppose you want to use the subgroup  $\{I, R^2\}$ . Figure out which element would have to be paired with  $FR^2$ .

	$\{I, R^2\}$			
$\{I, R^2\}$	$\{I, R^2\}$			
$\{FR^2 ?\}$				

Fig. 9. Coset formation task inspired by Rick's coset algorithm.

cards) for a failed partition and compare it to the operation table (also using colored index cards) for a successful partition in order to explain what differentiated this failed partition from the one that worked.

Teacher/Researcher: So here's your question. What's the difference between this situation [points to the operation table for the successful partition] and that situation [points to the partial table for an unsuccessful partition]?

Rick: This one we...

Sara: This one's symmetrical.

Teacher/Researcher: Symmetrical how?

Sara: It doesn't matter what order you put them.

Teacher/Researcher: The what, what are the 'thems'? Help me out with the pronouns.

Sara: The group and the  $g$  and  $H$ ,  $g+H=H+g$ . So I just wrote that before here, it must be that  $g+H=H+g$ .

While it was encouraging that the activity of comparing the successful partition to a failed partition seemed to support Sara in identifying normality as a necessary condition, there was insufficient evidence in the data to support any conjectures about why she saw this condition as necessary. Nevertheless, based on this, we designed a worksheet (Fig. 10) that invited students to compare a partial table for a failed partition with the successful partition. When we implemented this task during the whole-class teaching experiment, we were able to glean more insight into how it supported students in identifying this condition.

Dr. James: This worksheet has the full table for  $I R^2$  down here, this is the one that works. And then here's  $I$  paired with  $FR$ . It doesn't work and you're going to explore that table and see what goes wrong. So just take a few minutes and think about that individually.

Charles: I think that I got a pretty good explanation. The left coset and the right coset don't match.

Lee: Yeah.

Charles: Just use the one element and you get that. I used  $R$  and, I did  $RI$  and  $RFR$  and I got these, and then I did  $IR FRR$  and I got these two. So you can't have  $F$  be a part of this subset and  $FR^2$  be a part of, it doesn't work. You started generating a ton of elements

Lee: And it doesn't work out on the right. It creates the yellow-purple group. Instead of just...

Charles: You get like four elements instead of two, right?

Lee: Yeah.

Here the students are considering the subgroup  $H = \{I, FR\}$ . They then consider the subset that contains the element  $R$ . Multiplying  $R$  through in both directions, they generate three different elements (and then more elements after including



### Why does $\{I, R^2\}$ work and $\{I, FR\}$ not work?



Fig. 10. Normality worksheet.

any of these new elements as the second member of the coset containing  $R$ ). However, since  $H$  must work like the identity element, these products should produce only elements from the two-element coset that contains  $R$ . Thus they conclude that the subgroup  $H$  did not behave like an identity element because the right cosets did not equal the left cosets. This is elaborated in the following excerpt.

Kevin: I think that when you are first calculating this one you are using like, maybe, because you generate  $R$  then you generate the next one  $FR$ , multiply  $R$ .

Lee: Right, right

Kevin: But on the other hand if you put this way here its on the left hand side. So,  $R$  multiply  $FR$  is not  $FRR$ .

Lee: Right, its not.

Kevin: So that way it doesn't work.

Lee: Because the identity on the left has to be the identity on the right, so, yep.

Charles: Cosets don't match for  $R$ .

More insight into how students may think about this task can be gleaned from the whole-class discussion led by Dr. James after the students had worked for a few minutes in small groups.

Dr. James: Alright, so I think um, I think maybe some groups are ready to share what they found. So, let me just rephrase the question. Um, why is it that say  $IR^2$  seems to work but  $IFR$  doesn't seem to work. Anybody tell us something that they found that makes the one work but the other one not work?

What's the difference, yeah?

Jenna: It's not commutative, it's not a center.

Dr. James: Ok, so it's not the center. So how did that, why was, how did that come up to be important? I mean where.

Jenna: Identities need to be commutative.

Dr. James: Identities need to be commutative. Ok, can you just talk me through it a little more, because I wanna get this up here. So, does any one else want to second that? I mean it seems like everyone is nodding when you say that. So what does that mean that identity has to...

Jenna: The left coset should be the same as the right coset.

Dr. James: Ok, and how, so based on what? Like, I think everyone agrees –

Jenna: On the definition of identity.

Dr. James: So how do you get from the definition of identity to left coset has to equal the right coset?

Travis: Ah well if we're using this, these cosets to generate these groups. If you start with an element to generate on the left and the right, and if we compare this on the left and the right and they're not equal to each other then the new subset we generate is going to have three elements. Which would be too big.

At this point, another student, Ben, interjected<sup>14</sup> to address Jenna's initial suggestion that the identity subset (subgroup) needed to be the center of the group (which is the case for the two element normal subgroup of  $D_8$ ). He did so by reminding the class of an earlier example in which a quotient group construction was successful although the identity subset was not the center (or in the center).

<sup>14</sup> The center of a group is the subset (subgroup in fact) consisting of all elements that commute with every element of the group. This concept was introduced in an earlier unit in a standard exercise designed to provide students with practice with the subgroup concept.

Ben: Um, so that works good here, but we were looking at that too and if you do  $D_6$  and you look at the rotations right, that should work as the identity thing, the identity element. If you split it into two groups, you know what I'm talking about? So these are the rotations, and the flips, and the rotations and the flips.

Jenna: Yeah.

Ben: But the rotations aren't the center of  $D_6$ . So it kind of works in this case but it doesn't always work.

Jenna: It's a commutative though. You can, the left coset is the same as the right coset.

This discussion is very reminiscent of the proofs and refutations process described by Lakatos (1976) in his book on the historical process of mathematical discovery. Dr. James acknowledges this aspect of the discussion as he brought it to a close, characterizing it as a "point counterpoint."

Dr. James: Ok, so let me just time out here. This, this is amazing. So, this is great. So, um we have um a pretty good like point counterpoint happening right now. So, on the one hand it seems like the right coset has to equal the left coset. Um, which seems a little bit like commutativity. But then [Ben] raises the point that well, we had an example, namely the uh rotations of the triangle. That is a three-element subset of the six-element group, that does work, but we know that the rotations don't commute with everybody else in the group. So, it's not truly commutativity. And then the response to that was, well maybe element-by-element they don't commute but still the right coset equals the left coset. So it's like the coset itself sort of commutes.

After summarizing this debate, Dr. James introduced the term normal subgroup and defined it in terms of the equality of left and right cosets. In the whole class discussion excerpt, we see a good bit of the proofs and refutations process in action. Jenna conjectures that the subgroup needs to be the center of the group. She goes on to argue specifically that this condition is required because the subgroup needs to satisfy the identity property, and in particular it is a consequence of the fact that the identity must commute with every element of the quotient group. Travis elaborates, pointing out that in the example that does not work, when you multiply the subgroup on the right by an element (of one of the other subsets), you will get a different subset than when you multiply on the left (in fact you get a subset with more than two elements). Then yet another student steps in and offers a counterexample to Jenna's conjecture. Ben observes that in  $D_6$  the subgroup consisting of the rotations can be used to form a quotient group although this subgroup is not part of the center (which in  $D_6$  consists of only the identity element). Jenna then responds to this by observing that there is still a type of commutativity at play in that the left and right cosets are equal.

The identification of normality as a necessary condition is, in a sense, the end of the process of reinventing the quotient group concept, because all of the rules and procedures needed to successfully construct a quotient group have been established. In terms of the emergent models heuristic, at this point the quotient group concept is developed sufficiently to serve as a model for supporting more formal mathematical activity.

The proofs and refutations heuristic was key to developing all three aspects of the fourth step of the LIT and can be used to explain the students' progress. In each case, the instructional approach was to engage students in proving (or attempting to prove) that a partition formed a group under set multiplication. And, in each case students' reasoning about this proving activity supported them in developing an important aspect of the quotient group concept. As they worked to prove that a given (partial or complete) partition had an identity element, students developed the conditions that (1) one of the subsets (the identity subset) must be a subgroup, (2) the remaining subsets must be cosets of this subgroup (formed by multiplying an element by each element the subgroup), and (3) the subgroup must in fact be normal (its left and right cosets must be equivalent as sets). Each of these conditions was found to be necessary in order for a verification of the identity property to be successful.

## 6. Conclusions and implications

In this paper, we have presented a local instructional theory for supporting the guided reinvention of the quotient group concept. This local instructional theory takes as its point of departure the parity of the integers. Students begin by looking for analogs to evens and odds in a familiar finite group (in the instructional sequence presented here, this was the group of symmetries of a square). This notion of parity is then generalized to the idea of partitioning a group to form a group of subsets (under set multiplication). As this process unfolds, students begin to establish necessary conditions for such a construction to work – eventually developing the ideas that the identity subset must be a subgroup, that the partition must consist of cosets of this subgroup, and finally that the left and right cosets of this subgroup must be equivalent (normality). Our research suggests both that Burn (1996) was right to see promise in the fact that parity is (from an expert's perspective) an accessible example of a quotient group and that Dubinsky et al. (1997) was also correct in pointing out that understanding parity is much different from understanding the general quotient group concept. The LIT presented here describes a trajectory by which students can navigate the distance between this simple example and the general quotient group concept.

The local instructional theory can be framed in terms of the emergent models heuristic from Realistic Mathematics Education as well the proofs and refutations heuristic proposed by Larsen and Zandieh (2007). First, the quotient group concept emerges as a model of the students' informal activity as they partition the group of symmetries of a square into evens and odds. This model-of becomes more robust as students work to partition this group into a group consisting of four subsets. While the students work primarily in the context of the symmetries of a square, their activity looks toward the more general as they investigate whether properties observed in this context are necessary. For example, they prove that the identity subset *must* be a subgroup but that it need not be the center of the group. These aspects of the local instructional theory can be seen as closely related to the process of proofs and refutations described by Larsen and Zandieh

(2007). Conditions are evaluated as students consider proofs of their necessity in light of examples, non-examples, and counterexamples. Here, like Lakatos (1976), we consider a proof to be a kind of thought experiment. For example, the students imagine constructing (or actually begin constructing) a quotient group using a subgroup that is not normal and, by analyzing this process, realize that it will not work because the left and right cosets are not the same. Then (as we saw with Ben above) by considering a different example, the students can rule out a conjecture that the subgroup must be in the center of the group.

Finally, when the students have established a viable set of necessary conditions for partitioning a group into a quotient group, these conditions can be tested as students attempt to form other quotient groups or attempt to formally prove that they are necessary (and/or sufficient). For example, in the current version of the TAAFU quotient group unit, students prove that the cosets of a normal subgroup are closed under set multiplication ( $aHbH = abH$ ) and this activity can be leveraged to motivate a transition to multiplying cosets via representatives (as is common in standard treatments of the topic). This new version of the operation can be shown to be well-defined and then used to efficiently prove that normality is a *sufficient* condition for a set of cosets to form a group. This aspect of the TAAFU quotient groups LIT (the transition to more formal activity) is an area in which research is ongoing. In particular, we are continuing to investigate the question of how this more formal activity can best be supported in a way that stays connected to the students' informal activity.

The primary purpose of a local instructional theory is to support the design of an instructional sequence that is appropriate for a given instructional context. For example, an instructor teaching a group theory course in which students are more experienced with permutation groups than dihedral groups could construct an instructional sequence based on the LIT in which student partitioned  $S_3$  and  $A_4$ .<sup>15</sup> In this case, the instructor might want to start with  $S_3$  because it can be partitioned to form a quotient group of order 2 (evens/odds) while  $A_4$  cannot. The instructor may also have to adjust for the fact that neither of these groups has a pair of subgroups of the same size with the property that one is normal while the other is not – so comparisons to discover necessary conditions would involve partitions of different sizes. In this way, the LIT can be considered to be a generalized instructional sequence that can be adapted to various instructional contexts.

The TAAFU curriculum includes an instructional unit based on the LIT presented here that was created for an introductory group theory course. The curriculum materials are integrated with a set of interactive instructor support materials (Lockwood et al., in this issue) and can be found online at <http://www.web.pdx.edu/~slarsen/TAAFU/home.php>. A number of mathematicians have successfully used this curriculum and efforts are ongoing to study the efficacy of the curriculum related to supporting students' understanding of the quotient group concept. Early results from a preliminary comparison study have been encouraging in that students from TAAFU classrooms have performed significantly better on task-based quotient group questions included on an open-ended assessment instrument (see Larsen, Johnson, & Bartlo, in this issue).

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<sup>15</sup>  $S_3$  is the group (under composition) of permutations of a set of three elements, while  $A_4$  is a subgroup of the group of permutations of a set of four elements. Both are non-commutative and contain both normal and non-normal proper subgroups, so these groups could provide an appropriate starting point for reinventing the quotient group concept.

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