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University of Portland **Fall 2012**
Department of Mathematics **8/27-12/13**

MTH 441A: Modern Algebra I
MWF, 2:40-3:35 PM
Franz 212

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Webpage: We will use `learning.up.edu` for course updates (Moodle)
Office Hours: **M** 3:45-5:00, **T** 2:30-4:00, **W** 11:00-12:00, **F** 9:15-10:30, and by appointment

A Word From Your Instructor

Welcome to our class! I am looking forward to working with each of you this semester and will do everything in my power to ensure that our time together, both in class and out, is productive and enjoyable. I hope you will feel comfortable coming to me with any questions or concerns that arise, mathematical and otherwise, because I am here to help you learn. Not only is it my job, it is something I love doing, so please know that I am happy to assist you in any way I can.

This course is an introduction to what is a crucial and exciting area – algebra – which is a cornerstone in any modern mathematics program. Abstract (or modern) algebra is a pillar of mathematics; it is a field comprising elegant theory and deep results, and has far-reaching implications and connections to a bevy of other fields. Its development was spawned by great mathematical minds that included Euler, Gauss, Lagrange, Galois, Cauchy, Abel, Cayley, Hilbert, Artin, and Noether, among many others. Though its roots trace back to the early 18th century, it remains a lively and active field of research today, and lends itself to countless other fields of study: geometric and combinatorial group theory, representation theory, Lie theory, combinatorics, cryptography, algebraic topology, conservation laws in physics, and even crystalline structures in chemistry and materials science. The study of groups in particular will be the focus of the first semester of this two-course sequence, and while there is certainly more to explore than we will have time for, I hope you will gain a sense of the profound impact that the definition of an 'abstract group' has had on modern mathematics, as well as appreciate a number of important implications and examples.

Now, on to the nitty gritty. Most (if not all) of your questions about our course will be answered in the remainder of this syllabus. Please read it carefully before our next meeting.

Materials

Textbook: *Abstract Algebra: Theory and Applications*, by Thomas W. Judson (1997, Gnu Free Documentation License); available for purchase at the UP Print Shop. Also available online at `abstract/pugetsound.edu`.

Technology: For the most part, we will be using the traditional tools of mathematics: paper and pencil (or pen, if you choose). Part of the course objectives including learning to use the computer-based LaTeX typesetting package (see `http://www.tug.org/texlive/` or `http://www.tug.org/mactex/2011/`.

Course Content

Course Bulletin Description for MTH 441
An introduction to the study of the abstract algebraic structures that arise from generalizations of the natural arithmetic of the integers, polynomials over the integers, and rational numbers. Groups: examples, properties, and counting theorems. Rings: examples and properties. Fields: roots of polynomials and field extensions. *Prerequisite: MTH 311 and MTH 341*

Course Outline
The focus of this course is group theory and we will cover as many topics from Chapters 3 through 15 of the text as time permits. This includes examples and properties of groups, counting theorems, isomorphisms, homomorphisms, quotient groups, group actions, and Sylow theory. We will begin with a short introduction and motivation by way of examining symmetries of figures in the plane. Attempts will be made to supplement theory with background that highlights the historical context in which the field developed so that students can orient the material within the larger discipline of mathematics. The ultimate goal, in two semesters, is to cover the theory of groups, rings, and fields in reasonable detail and to finish with the Fundamental Theorem of Algebra.

Course Performance Objectives
This initial course in abstract algebra will provide students with the fundamental tools for studying the generalized algebraic structures underpinning group theory and understanding their relevance in mathematics at large. Very little of the course requires computational competency. As a course of mathematical theory, students will be familiar with the standard, classical results in algebra and are expected to learn to write complete, correct, and concise proofs of elementary results. In addition, students will increase their competence in conveying their results to their colleagues via in-class group work and discussion. Finally, students are will be able to typeset relevant mathematics using LaTeX .

Methods of Assessment

Students' mastery of the course objectives will be assessed by traditional means: written homework, quizzes, in-class activities, presentations, midterm exams, and a comprehensive final exam. The development of analytical and logical reasoning skills is inherent in the nature of mathematics and these skills are therefore assessed in conjunction with the course performance objectives.

Assessment Disclosure Statement: Student work products for this course may be used by the University for educational quality assurance purposes.

Attendance
You are expected to attend each class session with the understanding that your presence and participation are a vital part of the class community. You are responsible for material presented in class and noting any information or changes announced in class. In light of the emphasis in this course on collaborative in-class work, discussion, and presentations, unexcused absences and tardiness may adversely affect your final grade. It is also understood, however, that circumstances inevitably arise which might prevent one from attending class on occasion; communicate with me in advance whenever possible if you are forced to miss class for some reason, and I will work with you to stay current in the material.

Homework
Homework will be assigned weekly (more or less) and generally due two to three class meetings later. Some problems may be taken from your textbook but many will be of my own design. In some cases, I may invite you to revisit some problems from your homework after it has been graded in order to reinforce fundamental ideas and help you fine-tune your proof-writing skills, as these are both important goals of homework. I will indicate on your graded homework which problems, if any, you should consider rewriting for me to regrade.

Students are encouraged to work together on homework, as there are benefits to be found in discussing multiple perspectives on a problem and justifying one's thinking to others. All work turned in must be your own, of course. You are welcome to typeset your homework, though this will not be required for regular assignments. I encourage that you try to do this with at least some problems in your homework, though, as it allows great flexibility for rewriting and editing problems once you have feedback on them. Finally, we will not generally have time to discuss homework problems in class, so students are encouraged to visit me in office hours to work through issues in the material as they arise.

Quizzes/Class Activities
Short quizzes will be given every week or two throughout the semester, usually announced in advance. There are no make-up opportunities for missed quizzes or other in-class activities, unless excused by the instructor. Group activities will take place nearly every day in class, and this will be a crucial component of students' exposure to the course material. Conversations that take place in class are impossible to duplicate, but do check in with classmates or the instructor regarding missed content if you do miss a class.

Exams
There will be two midterm exams during the semester, and a comprehensive final exam. All exams will have a timed, in-class component but are also likely to involve a take-home portion (on which you are expected not to collaborate with others, and may have limited access to me). *Tentative* exam dates appear below in the grade breakdown; I will finalize the exam dates at least one week in advance of any exam.

There will be a cumulative final exam which will take place during the two-hour period as scheduled by the University. As per University policy, you will not be allowed to take the final exam at another time, except in the case of a University-sponsored absence, *about which you must notify me by the second week of class*. Please put the following on your calendar:

Our final exam is 10:30am –12:30pm, Tuesday, December 11th.

It is my policy not to give early exams or make-up exams. There are exactly two exceptions: an absence due to extreme hardship, or a University-sponsored event. In the latter case, you must inform me at least one week in advance, and within the first two weeks of the semester if you must miss the final exam. You may be asked to take the exam early. In case of extreme hardship (*e.g.*, illness or death of a family member), please notify me in advance if at all possible and I will work with you to make alternate plans. I reserve the right to deny make-up work and penalize absences that are not verified (via a health practitioner's note *e.g.*).

Course grade breakdown

Homework	15%
Quizzes and group activities	15%
Exam 1 (9/28)	20%
Exam 2 (11/9)	20%
Final Exam (12/11)	30%

Final grades in the course will be determined at the end of the semester, though will be no lower than those set forth in the following table.

A	90	–	100%
B	80	–	89%
C	70	–	79%
D	60	–	69%
F	0	–	59%

Getting Help

My sincerest advice to you: **do *not* wait until the last minute to get help.** Mathematics is a cumulative subject and material builds on prior knowledge. I have a number of office hours set during which you can stop by to get help on any problems. If you cannot make it to my office hours, make an appointment to see me for help. **I am here to help you, provided you are willing to seek and accept help!** If you email me with a question (which is fine), please be specific about the problem, what you've tried, and where you're stuck. Also, keep in mind that I may check my email less frequently than you would like, and rarely return email late at night. *Coming to visit my office is the most effective and efficient way to get help.*

The remainder of this document describes other resources available to you, as well as relevant aspects of the University's academic policy.

UNIVERSITY POLICIES & RESOURCES

The Learning Resource Center

The Learning Resource Center, located in Franz 120, houses the Writing Center, Math Resource Lab, Speech Resource Center, Group Process Assistance, and International Language Assistance (French, Spanish, German, Chinese).

The Writing Center is open by appointment and works to support professors and students as they write across the disciplines. Appointments are made electronically. To schedule an appointment, go the Writing Center website at `http://www.up.edu/lrc/writing/` and click on Appointments to sign in and view the schedule.

The Math Resource Lab offers tutoring to students studying mathematics Sunday through Thursday. Help with mathematics is available on a walk-in basis or, to schedule an appointment, call (503) 943-8157.

The Department of Communication Studies offers assistance to students at the University of Portland who seek to plan, prepare, practice, and deliver public presentations. Speech assistants are available by appointment only. To schedule an appointment go to `http://www.up.edu/lrc/speech` and click on the email link.

The Department of Communication Studies also offers assistance to students working on group projects. For information on how to schedule an appointment, go to `http://www.up.edu/lrc/groupprocess`.

The International Languages and Cultures Department offers tutoring support in French, German, Chinese, and Spanish. To make an appointment, go to `http://www.up.edu/lrc/languages/signup`.

Withdrawal Proceedures: It is the student's responsibility to drop the course if he or she is no longer planning on attending the course or filling the other course requirements. In order to drop, the student must use and Add/Drop form available at the Registration Office. If a student does not properly withdraw from a course, he or she may receive an **F** for the course. A properly withdrawn student will receive a **W**. The last day to withdraw is **Friday, November 16th.**

Incompletes: An incomplete **(I)** may be given when the quality of a student's work is satisfactory (C or better), but for some essential reason the course has not been completed by the student. An (I) is reserved for emergency situations only. To request an incomplete, the student needs a typed, signed and dated letter stating the reason(s) why an incomplete is appropriate. The letter should also contain the conditions for the completion of work. Acceptance of the request shall be at the instructor's discretion.

Accommodation for Disability
If you have a disability and require an accommodation to fully participate in this class, contact the Office for Students with Disabilities (OSWD), located in the University Health Center (503-943-7134), as soon as possible. If you have an OSWD Accommodation Plan, you should make an appointment to meet with me to discuss your accommodations. Also, you should meet with me if you wish to discuss emergency medical information or special arrangements in case the building must be evacuated.

University of Portland's Code of Academic Integrity
Academic integrity is openness and honesty in all scholarly endeavors. The University of Portland is a scholarly community dedicated to the discovery, investigation, and dissemination of truth, and to the development of the whole person. Membership in this community is a privilege, requiring each person to practice academic integrity at its highest level, while expecting and promoting the same in others. Breaches of academic integrity will not be tolerated and will be addressed by the community with all due gravity (*taken from the University of Portland's Code of Academic Integrity*).
The complete Code may be found in the current University of Portland Student Handbook, as well as the Guidelines for Implementation. It is each student's responsibility to inform him or herself of the Code and Guidelines.

WEEK 1

DAY 1: Groups L1 T1-5

- Intro/motivation
- T2 & T3: Discussed two ways to consider symmetry (combined T2 and T3). Gave examples of isometry of plane vs. isometry of a figure in the plane.
- T4: **Handout (Groups, L1T4)**. Also gave each group cutout X.
- T4 & T5: *Figure X symmetries; how do we count?* Begs the question: when are two symmetries equivalent? Discussion about path independence. Students had examples of 4, 8, and 22 symmetries.

DAY 2: Groups L1 T5-8, L2 T1-4

- Review T5.
- T6-T7: *How do we define equivalence?* Revisit defn of symmetry (a function), settle on functional equivalence. (Combined T6 and T7).
- *Find symmetries of an equilateral triangle.* Confirmed.
- *Justify there can be no more than 6.* Discussed at board. *Corollary:* If we combine any two of the 6 symmetries we already found, the result must also be a symmetry (one we already have!).
- *Agree on notation: let F, R denote....* **Handout 1 (Groups: Handout 1)** with triangles; students worked in groups. Found multiple ways to write each symmetry wrt R and F.

DAY 3: Groups L2 T4-6

- *In pairs: decide how to express each symmetry in terms of R and F.* Students generated all three types of notation (additive, multiplicative, function compositional), and found multiple ways to represent each symmetry wrt R and F.
- Students from each pair put answers on **overhead of Handout 1**, we voted on favorite notation (mult.), favorite representative for each symmetry. Talked about rules/identities that were used to verify multiple representations give the same map. Talked about associativity.
- Handed out **blank 6x6 table**; figure out which canonical symmetry each pair of symmetries represents. Students finished this in pairs.

WEEK 2

DAY 4: Groups L3 T1-5, L4

- T1: skip. Repeat of L2 T6.
- T2: *How did you fill out 6x6 grid - manipulating triangle or using rules to reduce?* Combination!
- Put up **overhead with 6x6 table** filled with unreduced symmetries. Pick specific boxes within 6x6 grid and have each group figure out the rules used in that box to transform the composition of two symmetries into a canonical one. Have students share calculations. Then, I listed possible rules used (based on our original task where we wrote symmetries in as many ways as possible). Only ones used were $F^2 = 1$, $R^3 = 1$, and $FR = RF^2$ (plus identity and associativity).
- T4: *Do you think we need all these rules, or are some extra?* Extra! Prove it:
- T5: *Given this list of 5 rules (dihedral relations), show that 2-5 can all be derived from 1.* Have each group put up one calculation.
- Theres still one rule that our set of symmetries exhibits, well get at it by undertaking the following:
- L4 T1: *Prove the Sudoku property*, **handout** (sudoku.pages). Had a group present Part I.

Sample Proof: Assume toward contradiction that B appears twice in row A. This means that, for some symmetries C and D, we have $B = AC$ and $B = AD$, hence $AC = AD$. (Justify that inverses exists and are symmetries.) Applying A^{-1} to each side, $C = D$ so B must only appear once in row A after all.

- **HW 1: Symmetries** assigned. Encouraged students to try Problem 3 before next class (symmetries of a square).

DAY 5: Groups L4, L5 T1-?

- L4: Sudoku property, Part II. Discuss. Consider difference between generic proofs and ones that rely on the finiteness properties of this particular set.

  Sample Proof 1: There are 6 symmetries and none can appear more than once (Part I). Thus, the Pigeonhole Property implies that each of the 6 entries in each row or column is filled with a distinct symmetry, so all symmetries appear at least once in a row or column.

  Sample Proof 2: Suppose toward contradiction that B does not appear anywhere in row A. By virtue of how we compute the entries in row A, this means that for all symmetries C, $B \neq AC$. Once again applying inverses (justified above), we have that for all C, $A^{-1}B \neq C$. This means that the product of two symmetries, A^{-1} and B, is not equal to any of the original symmetries, and is therefore not a symmetry. Contradiction.
- L5 T1: *find symmetries of non-square rectangle, and rotational symmetries of square.* Had students put on the board.

DAY 6: Groups L5 T2-4

- *What are the rules that were common to each of the examples we considered?* Discuss.
- List three axioms (plus some extras: Sudoku, commutativity, etc.). Brainstorm other 'systems' that satisfy these axioms. Students came up with $(\mathbb{R}, +)$, $(\mathbb{R}^*, \times)$, same for $\mathbb{Q}$, $(\mathbb{Z}, +)$, vector spaces, (matrices, +), (invertible matrices, $\times$), others.
- Ended with definition of group!

WEEK 3

DAY 7: Groups L6

- Chat about modular arithmetic

DAY 8: Groups L

- **HW #2: Practice with Groups** assigned.

DAY 9: Practice

- Hannah subbed: Identifying Groups handout. They got through most or all, and discussed many as a class.

WEEK 4

DAY 10: Groups L7 T1-4

- Gave all tasks at once. T1: Define *order* of an element and calculate order of all elements in D_8.
- T2: Given info about order, prove things about relations between group elements.

- T3: Prove G is Abelian if and only if ...
- T4: Make a conjecture about what $(ab)^{-1}$ is in general, and prove your answer.

DAY 11: Groups L7 T1-6 (recap), L8 T1-6

- Go over T3, T4 from last time, and last few examples on the Identifying Groups handout.
- Added a task, T5: Are inverses unique? Prove your answer.
- T6: skipped (practiced last week with worksheet).
- Start subgroups (L8).
- T1: *Show* $5\mathbb{Z}$ *is a subgroup of* $\mathbb{Z}$. Went quickly.
- T2: *Come up with a weaker set of conditions that guarantee a subset is a subgroup.* Class conjectured conditions that we needed to check, and almost immediately ruled out closure as being sufficient (T5). Then, noticed that we didn't need identity condition if we had inverses. Also discussed that the identity and the inverses in H had to be the same as they were in G, as the operation isn't changing (T3). Hence, skipped T3 and T5 (formally).
- T3: *Prove Subgroup Criteria Theorem*. Did as a class on the board.
- T6: skip. Already discussed with counterexample.
- **HW #3**: Groups and Subgroups assigned.

DAY 12: Groups L8 T7

- Remind them of our 'Testing for Subgroups' Theorem from last time, and add a task to prove an equivalent theorem:
- T7 (new): *Prove that H is a subgroup of G iff H is nonempty and whenever $g, h \in H$, gh^{-1} is also in H.*
- Move into lecture: Examples of groups; cyclic groups and subgroups.

WEEK 5

DAY 13: Examples of Groups

- Cyclic groups and subgroups, continued.

DAY 14:

- More on cyclic groups.
- Application: Multiplicative groups of complex numbers with cyclic subgroups (n^{th} groups of unity, etc.)
- ***Quiz*** over groups and subgroups.

DAY 15: Isomorphisms L1 T1

- Finish complex cyclic groups.
- Start Unit 2 L1: *Decide if Mystery Group is D_6 or not*.
- Students figured out, yes, it's the same as $D_6 = S_3$ and put their correspondences on the board (there were two).
- Conjecture: (Sean?) It doesn't matter what elements you assign to others, as long as they have the same order.

WEEK 6

DAY 16: EXAM 1 REVIEW

- Went over HW and Quiz.
- Review WS.

DAY 17: EXAM

- EXAM 1 (in-class portion).
- Too long! Asked students to take home and finish what they were working on for partial credit.

DAY 18: ISOMORPHISMS L2 T1 - T3

- Take-home due.
- Iso L2 T1: *Explain why the following correspondence doesn't work to show that the Mystery Group is D_6.* In the process, found a counterexample to conjecture from last time: it's not only order of elements that matters, it's how we compose elements.
- New Conjecture: Maybe permuting all elements of a certain order will preserve a working correspondence.
- L2 T2: *Complete the following map between Mystery Group and D_6.* Students again focused on order of elements to argue why D had to equal R^2 (because it's the only thing left of order 3), rather than relying on Mystery Group table to argue that $G^2 = D$ and if $G = R$ then $D = R^2$.
- L2 T3: *Finish the following definition: "Groups $(G, \circ)$ and $(H, *)$ are* **isomorphic** *if . . . ".* Many suggested that we need to map elements to other elements of the same order, but we reduced this to making sure operation works. Finished by putting the definition on the board.

WEEK 7

DAY 19: Isomorphisms L3 T1-T3

- T1: *Recall the definition of* **isomorphism**. [Handout: Iso L3.]
- T2: *Is the group given by the following table isomorphic to C_4 (rotations of the square) or V_4 (Klein 4-group)?* Supplemental notes on this one – explain names of groups, and state Lagrange's Thm so students can (on HW) prove that there are exactly two groups of order 4, up to isomorphism.
- T3: show isomorphisms preserve inverses and commutativity.
- Intermediate lemma: Isomorphisms preserve the identity.

DAY 20: Isomorphisms L3 T3-T6

- T4: Induction proof that isos preserve powers of elements.
- T5: Show isos preserve cyclic behavior.
- T6: Show isos preserve order.

DAY 21:

- Students present facts about isomorphisms. (T1-T6)

FALL BREAK

WEEK 8

DAY 22: Isomorphisms lecture

- Recap of what we know about isomorphisms.
- Classification of infinite and finite cyclic groups ($\mathbb{Z}$ and $\mathbb{Z}_n$).
- Cayley's Theorem: *Every finite group of order n is isomorphic to a subgroup of S_n.* [Handout: quaternion]

DAY 23: Isomorphisms lecture

- Review Cayley's Thm and finish example (needed for HW where they use method of Cayley's to embed V_4 as a subgroup of S_4)
- External direct products
- $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n \iff m$ and n are relatively prime.

DAY 24: Isomorphisms lecture

- Fundamental Theorem of Finite Abelian Groups
- Internal direct products

WEEK 9

DAY 25: Quotient Groups

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DAY 26: Quotient Groups

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DAY 27: Quotient Groups

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WEEK 10

DAY 28: Quotient Groups, Lesson 2

- T?-T?:
- Definition of coset; find L and R cosets of the subgroups $H_1 = \{1, R^2\}$ and $H_2 = \{1, FR\}$. Do either of these partitions form a quotient group?

DAY 29: Exam Review

- Handout: Exam 2 Practice
- Go over old HW / quiz

DAY 30: EXAM II

- Covers isomorphisms, direct products, Cayley's Thm, Lagrange's Thm, Fundamental Thm of Finite Abelian Groups

WEEK 11

DAY 28: Quotient Groups, Lesson 3

- T1: recap what we have discovered about forming quotient groups.
- T2: definition of **normal**.
- T3: new notation for quotient G/H. **Thm:** G/H is a group $\iff$ H is a normal subgroup of G.
- T4: Prove that if G/H is a group, then $Hg \subseteq gH \forall g \in G$. (Got started, but stuck.)

DAY 29: Quotient Groups

- T4: Prove that if G/H is a group, then $Hg \subseteq gH \forall g \in G$.
- T5: Prove that if $Hg \subseteq gH \forall g \in G$, then $ghg^{-1} \in H$ for all $g \in G$.
- T6:
- Still a fair amount of confusion about what the elements in G/H *are*.

DAY 30: Quotient Groups

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EXTRA TOPICS

(1) **Groups**
   (a) Permutation groups (S_n and A_n) via HW
   (b) Cyclic groups and subgroups (including $n\mathbb{Z}$, $\mathbb{Z}_n$)
   (c) Complex groups (circle group, $\mathbb{T}$) and cyclic subgroups (roots of unity)

(2) **Isomorphisms**
   (a) Lagrange's Theorem (in order to prove there are exactly two groups of order 4, up to isomorphism). Stated (for now) without proof.
   (b) Cayley's Theorem: *Every group is isomorphic to a group of permutations.* (Proved via example)
   (c) Direct products of groups, internal and external.
   (d) Fundamental Thm of Finite Abelian Groups (stated without proof)

(3) **Quotient Groups**
   (a) (Cosets, homomorphisms, normal subgroups)
   (b) Isomorphism Theorems (Chp 11)
   (c) Classifying finite Abelian groups (Chp 13)

Homework 3:
Groups and Subgroups

Due: Wednesday, 9/26

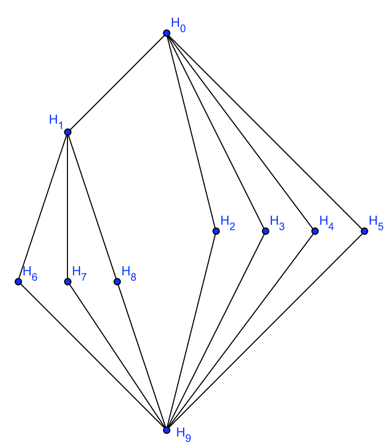
(1) Suppose G is a group with $a^2 = \mathbf{e} = b^2$ and $ab = ba$, but $a \neq \mathbf{e}$, $b \neq \mathbf{e}$, and $a \neq b$. Show that $H = \{\mathbf{e}, a, b, ab\}$ is a subgroup of G.

(2) The **center** of a group, G, is defined to be the subset consisting of all elements $c \in G$ such that $cx = xc$ for all $x \in G$ (*i.e.*, the elements that commute with everything in G). The center of G is usually denoted $Z(G)$. Prove that $Z(G)$ is a subgroup of G.

(3) Let a be any element of a group G and let H be the subset consisting of all elements of the form a^n where n is an integer. Show that H is a subgroup of G.

(4) Recall that the **general linear group**, $GL_n(\mathbb{R})$, is the set of invertible $n \times n$ matrices with real entries.
   (a) Show that the set $SL_n(\mathbb{R})$ of $n \times n$ matrices with determinant equal to 1 is a subgroup of $GL_n(\mathbb{R})$. $SL_n(\mathbb{R})$ is called the **special linear group**.
   (b) Prove or disprove: $SL_n(\mathbb{Z})$, the set of 2×2 matrices with entries in $\mathbb{Z}$ and determinant 1, is a subgroup of $SL_n(\mathbb{R})$.

(5) Recall that S_n, the **symmetric group on** n **letters**, is the group of permutations of the set $\{1, 2, \ldots, n\}$.
   (a) Read Chapter 5 (Permutation Groups) in your text, paying particular attention to the definitions of **transpositions**, and **even** and **odd** permutations. Understand the proof that A_n, the **alternating group on** n **letters**, is a subgroup of S_n.
   (b) Express the following cycles as products of transpositions and identify each as even or odd.
   (i.) (14653) (ii.) (16732)(1423)(43256) (iii.) (234)(165) (iv.) (143)(1436)
   (c) List all the elements in A_4. (Note the number of elements in S_4 and the number of elements in A_4.)
   (d) Find all subgroups of A_4 and note the **order**, or number of elements, of each subgroup.
   (e) Determine where each subgroup of A_4 belongs in the **subgroup lattice** pictured below. Subgroups H_i and H_j are joined by one or more line segments in the lattice if and only if H_i is a proper subgroup of H_j.



EXAM 1

IN-CLASS PORTION

PART A

Answer both of the following questions in the blue book provided. Please justify your answers briefly but carefully.

(1) ***True or False?*** Decide if each statement is true or false, and give a brief justification or counterexample to support your answer. *(5 pts each)*

(a) $\mathbb{Z}_6^*$ is a group under the operation of multiplication.

(b) If a nonempty subset H of a group G is closed under the operation of G, then H is a subgroup of G.

(c) There exists a group of order n for any positive integer n.

(d) $\mathbb{Z}$ is a group under the operation of subtraction.

(2) Let G be the group of 2×2 real-valued matrices under the operation of addition and let

$$H = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a + d = 0 \right\}.$$

Prove that H is a subgroup of G. *(20 pts)*

PART B

Pick 3 of the following 5 questions and indicate clearly in your blue book which problems you have chosen to answer. Fully justified thinking and complete proofs are expected in each case. Each question is worth a total of 20 points.

(1) Suppose that G is a finite group.
   (a) Finish the following definition: "G is **Abelian** if ..."
   (b) Prove that if every non-identity element in G has order 2, then G is Abelian.
   Extra Credit: Use the above result to explain why, structurally speaking, there must be exactly two groups of order 4.

(2) Suppose that H and K are subgroups of a group G.
   (a) Prove that the intersection $H \cap K$ is a subgroup of G.
   (b) Find a counterexample that demonstrates the union $H \cup K$ is not necessarily a subgroup of G.
   Extra Credit: Determine necessary and sufficient conditions for $H \cup K$ to be a subgroup.

(3) Let $G = \mathbb{Z}_{20}$.
   (a) Determine all generators of G.
   (b) Determine all subgroups of G and explain how you can be certain you have found all of them.
   (c) Sketch a subgroup lattice of G.
   Extra Credit: Determine the group of units of G, also denoted $U(20)$, and build its operation table.

(4) Let G be a group.
   (a) Define the center $Z(G)$ of G.
   (b) If $x \in Z(G)$ has order 4 and $y \in G$ has order 5, prove that xy has order 20.
   (c) Suppose instead that $x \in Z(G)$ has order 4 and $y \in G$ has order 4. What is the order of xy?
   Extra Credit: Suppose $x \in Z(G)$ has order m and $y \in G$ has order n. Find the order of xy.

(5) Let H be a subgroup of G and define
$$N(H) = \{g \in G \mid gh = hg \text{ for all } h \in H\}.$$
   (a) Prove that $N(H)$ is a subgroup of G. (N stands for the *normalizer* of H in G.)
   (b) If $H = Z(G)$, where $Z(G)$ denotes the center of G, what is $N(H)$?
   Extra Credit: Compute the normalizer of the subgroup $H = \langle(123)\rangle$ in S_3.

EXAM 1
TAKE-HOME PORTION

Please write or type your solutions to the following problems (on separate paper); they are due Friday, 10/5 at the beginning of class. You may refer to your class notes and your book during this portion of the exam (and possibly a calculator), *but you may not consult any other source of any type*.

Please initial that you agree to adhere to the conditions for this take-home exam: _____________.

(1) Let $\alpha = (12345)$ and $\beta = (12)$ be elements of a subgroup H in S_5. You will determine the entire subgroup H by showing the following.

Note: the following composition of cycles is written in multiplicative notation, rather than in the function compositional notation your book uses.

- (a) Deduce that $(23) \in H$ by computing $\alpha^{-1}\beta\alpha$.
- (b) Deduce that $(34) \in H$ by computing $\alpha^{-1}(23)\alpha$.
- (c) Show that (45) and (15) belong to H.
- (d) Show that $\gamma = (23)\beta(23)$ belongs to H.
- (e) Show (24), (35), (14), and (25) also belong to H.
- (f) Finally, making use of Proposition 5.4 in your book, deduce a simple way to describe the subgroup H.

(2) Let

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

be elements in $GL_2(\mathbb{R})$.

- (a) Show that A and B each have finite order.
- (b) Show that AB does not have finite order.
- (c) Explain how parts (a) and (b) demonstrate that $A^nB^n \neq (AB)^n$ in $GL_2(\mathbb{R})$. What else can you deduce about the group from this fact? (*Hint:* think about the $n = 2$ case...)

  Bonus: Explain geometrically (in terms of linear transformations of the plane, for example) why A above has finite order. Can you do the same for B?

EXAM 2
IN-CLASS PORTION

PART A

Answer **both** of the following questions in the blue book provided. Please justify your answers briefly but carefully. Each question is worth a total of 20 points.

(1) ***True or False?*** Decide if each statement is true or false, and give a brief justification or counterexample to support your answer.
   (a) $S_3 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$
   (b) If n divides $|G|$, then G has an element of order n.
   (c) If $p \geq 3$ is prime, then any two Abelian groups of order $2p$ are isomorphic.
   (d) A group with 3 elements has exactly 2 automorphisms.

(2) Prove that if G is any cyclic group of order n, then $G \cong \mathbb{Z}_n$.
(Please construct the isomorphism rather than simply citing a theorem.)

$\boxed{\text{PART B}}$

Pick 3 of the following 5 questions and indicate clearly in your blue book which problems you have chosen to answer. Fully justified thinking and complete proofs are expected in each case. Each question is worth a total of 20 points.

(1) Consider the groups $\mathbb{Z}_6$ and $\mathcal{U}(7)$, the group of units of $\mathbb{Z}_7$ (also denoted $\mathbb{Z}_7^*$).
   (a) List the elements in $\mathcal{U}(7)$. What is the group operation in $\mathcal{U}(7)$? What is the group operation in $\mathbb{Z}_6$?
   (b) Prove that $\mathbb{Z}_6 \cong \mathcal{U}(7)$.

(2) Let G be a group.
   (a) Give the formal definition of an *automorphism* of G.
   (b) Prove that if the map $\phi : G \to G$ defined by $\phi(x) = x^{-1}$ is an automorphism, then G is Abelian.

(3) Let $G = \mathbb{Z}_{10}$.
   (a) Prove that G is isomorphic to the internal direct product of subgroups $H \times K$, where $H = \{0, 5\}$ and $K = \{0, 2, 4, 6, 8\}$.
   (b) By finding more familiar groups that are isomorphic to H and K, write $\mathbb{Z}_{10}$ as an (external) direct product of two different groups.

(4) Let $G = \langle a \rangle$ and $H = \langle b \rangle$ be cyclic groups of order 6.
   (a) Complete the following table so that α and β are isomorphisms from G to H.

x	$\mathbf{e}$	a	a^2	a^3	a^4	a^5
$\alpha(x)$		b				
$\beta(x)$		b^5				

   (b) Explain why these are the only two isomorphisms from G to H.

(5) Suppose G is an Abelian group of order 450.
   (a) List the all the possible isomorphism types for G. Explain briefly how you know these isomorphism types are all distinct.
   (b) What is the largest possible order of a cyclic subgroup of the group $H = \mathbb{Z}_{12} \times \mathbb{Z}_{15}$? Explain your answer.

EXAM 2

TAKE-HOME PORTION

Please write or type your solutions to the following problems (on separate paper); they are due Monday, 11/12, at the beginning of class. You may refer to your class notes and your book during this portion of the exam (and possibly a calculator), *but you may not consult any other source of any type*.

Please initial that you agree to adhere to the conditions for this take-home exam: _____________.

(1) Let $GL_2(\mathbb{Z}_2)$ be the group of invertible 2×2 matrices with entries in $\mathbb{Z}_2 = \{0, 1\}$.

(a) List the elements of $GL_2(\mathbb{Z}_2)$. *(Hint: they must be* invertible!*)*

(b) Show that every element in $GL_2(\mathbb{Z}_2)$ can be written as a product of the matrices

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(c) Prove that $GL_2(\mathbb{Z}_2)$ is isomorphic to the symmetric group S_3.
(Hint: every element in S_3 can be written as a product of the permutations (123) and (13).)

(2) Recall that an *inner automorphism* of a group G is an automorphism of the form $x \mapsto gxg^{-1}$ where g is a fixed element in G. The set of all inner automorphisms of G is

$$Inn(G) = \{\phi_g \mid \phi_g(x) = gxg^{-1} \text{ for } x \in G\}.$$

(a) In HW, you verified that an inner automorphism is indeed an automorphism, so that $Inn(G)$ is a subset of $Aut(G)$, where $Aut(G)$ is the group of all automorphisms of G. Now prove that $Inn(G)$ is a subgroup of $Aut(G)$. (The group operation in $Aut(G)$ is function composition.)

(b) Give an example of a group, G, for which $Inn(G) = \{1\}$ and explain your example.

(c) Find and tabulate the inner automorphisms of S_3 (*i.e.*, write down explicitly where each inner automorphism sends elements of S_3, perhaps using a table similar to one you constructed in HW). Are there automorphisms of S_3 other than the inner automorphisms? How do you know?

Bonus: What's purple and commutes?

EXAM 2
TAKE-HOME PORTION

Please write or type your solutions to the following problems (on separate paper); they are due Monday, 11/12, at the beginning of class. You may refer to your class notes and your book during this portion of the exam (and possibly a calculator), *but you may not consult any other source of any type*.

Please initial that you agree to adhere to the conditions for this take-home exam: ____________.

(1) Let $GL_2(\mathbb{Z}_2)$ be the group of invertible 2×2 matrices with entries in $\mathbb{Z}_2 = \{0, 1\}$.
  (a) List the elements of $GL_2(\mathbb{Z}_2)$. *(Hint: they must be* invertible!*)*
  (b) Show that every element in $GL_2(\mathbb{Z}_2)$ can be written as a product of the matrices
$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$
  (c) Prove that $GL_2(\mathbb{Z}_2)$ is isomorphic to the symmetric group S_3.
  (Hint: every element in S_3 can be written as a product of the permutations (123) and (132).)

(2) Recall that an *inner automorphism* of a group G is an automorphism of the form $x \mapsto gxg^{-1}$ where g is a fixed element in G. The set of all inner automorphisms of G is
$$Inn(G) = \{\phi_g \mid \phi_g(x) = gxg^{-1} \text{ for } x \in G\}.$$
  (a) In HW, you verified that an inner automorphism is indeed an automorphism, so that $Inn(G)$ is a subset of $Aut(G)$, where $Aut(G)$ is the group of all automorphisms of G. Now prove that $Inn(G)$ is a subgroup of $Aut(G)$. (The group operation in $Aut(G)$ is function composition.)
  (b) Give an example of a group, G, for which $Inn(G) = \{1\}$ and explain your example.
  (c) Find and tabulate the inner automorphisms of S_3 (*i.e.*, write down explicitly where each inner automorphism sends elements of S_3, perhaps using a table similar to one you constructed in HW). Are there automorphisms of S_3 other than the inner automorphisms? How do you know?

Bonus: What's purple and commutes?

Practice for Final Exam

The final exam will be comprehensive, and is designed to measure :

- your understanding of the major areas of group theory: groups, subgroups, quotient groups, and the maps between these sets (isomorphisms, automorphisms, and homomorphisms);
- your fluency in the variety of examples of groups we explored: groups of symmetries and dihedral groups, symmetric (S_n) and alternating groups (A_n), matrix groups, finite/infinite groups, cyclic/non-cyclic groups, special subgroups (center, normalizer), etc.;
- and the major theorems about such groups: Lagrange, Cayley, Fund Thm of Finite Abelian Groups, Sylow's theorems, the various isomorphism theorems

All topics we explored in class and via homework/quiz/exam questions are fair game, but you should try to discern the major ideas in the course and focus on those. The material from class roughly corresponds to the following chapters of our textbook, which is an excellent reference for your studies: Chapters 3, 4, 5, 6, 9, 10, and 11, with selected topics appearing in Chapters 12, 13, and 15 as well.

Appearance of topics on the final will roughly correspond to the time we spent on them in class, with slight emphasis on those things over which you have not yet been tested (quotient groups and homomorphisms).

---

(1) Determine which of the following maps are homomorphisms. For each homomorphism, determine its kernel and image.
  (a) $\phi : \mathbb{Z} \to \mathbb{R}$ (under addition) defined by $\phi(n) = n$.
  (b) $\phi : \mathbb{R} \to \mathbb{Z}$ (under addition) defined by $\phi(x) =$ the greatest integer $\leq x$.
  (c) $\phi : \mathbb{Z}_6 \to \mathbb{Z}_2$ defined by $\phi(x) =$ the remainder of x when divided by 2.
  (d) $\phi : \mathbb{R}^* \to GL_2(\mathbb{R})$ defined by $\phi(x) = \begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix}$.
  (e) $\phi : \mathbb{R} \to GL_2(\mathbb{R})$ defined by $\phi(x) = \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}$.
  (f) $\phi : GL_2(\mathbb{R}) \to \mathbb{R}$ defined by $\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d$.
  (g) $\phi : GL_2(\mathbb{R}) \to \mathbb{R}^*$ defined by $\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$.

(2) We say a group G is **generated** by a set of elements $\{g_1, g_2, \ldots, g_n\}$ if every element $a \in G$ can be written as a composition of these elements (under whatever operation is specified in the group). For example, in the take-home portion of your last exam you showed that the group $GL_2(\mathbb{Z}_2)$ was generated by matrices A and B, and that S_3 was generated by the permutations (123) and (13).

  Explain why a homomorphism $\phi : G \to H$ (and therefore an isomorphism and automorphism) is entirely determined by where ϕ sends the generators of G.

(3) Describe all the homomorphisms from $\mathbb{Z}_{24}$ to $\mathbb{Z}_{18}$.

(4) Define $\phi : \mathbb{R} \to \mathbb{C}^*$ by $\phi(x) = \cos x + i \sin x$. Show ϕ is a homomorphism, find its kernel and image, and apply the 1^{st} Isomorphism Theorem to obtain a description of a familiar group as a quotient of other familiar groups.

(5) In the group $\mathbb{Z}_{24}$, let $H = \langle 4 \rangle$ and $N = \langle 6 \rangle$.
  (a) List the elements in HN (though with additive groups we usually write this as $H + N$) and $H \cap N$.
  (b) List the cosets in HN/N, showing the elements in each coset.
  (c) List the cosets in $H/(H \cap N)$, showing the elements in each coset.
  (d) A map between HN/N and $H/(H \cap N)$ can be defined by $\phi : H \to HN/N$ where $\phi(h) = hN$. Verify that $\text{Ker}\phi = H \cap N$ and in doing so, describe the correspondence between HN/N and $H/(H \cap N)$ guaranteed by the Second Isomorphism Theorem.

(6) (Sylow) Show that every group of order 45 has a normal subgroup of order 9.

(7) (Sylow) A **simple** group is a group containing no nontrivial normal subgroups. Show that a group of order 20 cannot be simple.

(8) Fill in the blanks:
  (a) The quotient group $\mathbb{Z}_6/\langle 3 \rangle$ is order _____.
  (b) The quotient group $(\mathbb{Z}_4 \times \mathbb{Z}_{12})/\langle (2,2) \rangle$ is order _____.
  (c) The coset $5 + \langle 4 \rangle$ has order _____ in the quotient group $\mathbb{Z}_{12}/\langle 4 \rangle$.

(9) Let G be a cyclic group and N be a normal subgroup of G.
  (a) Prove G/N must be cyclic.
  (b) By way of example, show that a quotient group G/N may be cyclic even if the group G is not.

(10) Find all the Abelian groups of order 200 up to isomorphism.

(11) Let G be a group and let $G' = \langle aba^{-1}b^{-1} \rangle$. That is, G' is the subgroup of all products of elements in G of the form $aba^{-1}b^{-1}$. The subgroup G' is called the **commutator subgroup** of G.
  (a) Show that G' is a normal subgroup of G. (It might also be good practice to verify that G' is a subgroup in the first place.)
  (b) Let N be any normal subgroup of G. Prove that G/N is Abelian if and only if N contains the commutator subgroup of G.

(12) Suppose that $[G : H] = 2$. If a and b are not in H, prove that $ab \in H$.

FINAL EXAM
IN-CLASS PORTION

PART A

Answer **all three** of the following questions in the blue book provided. Please justify your answers briefly but carefully. Each question is worth a total of 20 points.

(1) ***True or False?*** Decide if each statement is true or false, and give a brief justification or counterexample to support your answer.
   (a) Every finite group G with prime order p is cyclic.
   (b) An additive group cannot be isomorphic to a multiplicative group.
   (c) If H is a subgroup of G such that $x \notin H$ and $y \notin H$, then $xy \notin H$.
   (d) If N is a normal subgroup of G and $gN = N$ for some $g \in G$, then $g = e_G$.
   (e) $3\mathbb{Z} \times 6\mathbb{Z} \cong \mathbb{Z} \times \mathbb{Z}$.

(2) Consider the group G defined by the following operation table:

*	**A**	**B**	**C**	**D**
A	D	C	B	A
B	C	A	D	B
C	B	D	A	C
D	A	B	C	D

G must be isomorphic to one of two groups. Name or otherwise identify these groups and prove that G is isomorphic to one of the two groups you named.

(3) (a) Let G be a group. Give a definition of a **normal** subgroup of G. (You may state either our original definition or the one we later proved was equivalent.)
   (b) Let n be a fixed integer and consider $H = \{g^n \mid g \in G\}$. Prove that H is a normal subgroup of G. (You may find the equivalent definition more useful than our original here.)
   (c) Explain briefly why "normality" is an important property for a subgroup to have.

PART B

Pick 3 of the following 5 questions and indicate clearly in your blue book which problems you have chosen to answer. Fully justified thinking and complete proofs are expected in each case. Each question is worth a total of 20 points.

(1) (a) State the **Fundamental Theorem of Finite Abelian Groups**.

(b) Prove that $\mathbb{Z}_8 \not\cong \mathbb{Z}_2 \times \mathbb{Z}_4$.

(c) Classify, up to isomorphism, all Abelian groups of order 756 (where $756 = 2^2 3^3 7$). Your solution should refer to your argument above (or similar thinking) to justify that each isomorphism type you list is indeed unique.

(2) (a) State the definitions of a group **isomorphism** and a group **automorphism**.

(b) State the definition of $Z(G)$, the **center** of a group G.

(c) Prove that if $\phi : G \to H$ is an isomorphism, then $Z(G) \cong Z(H)$.

(3) Suppose G and H are groups.

(a) Give the definition of a **homomorphism** from G to H.

(b) Suppose $\phi : G \to H$ is a homomorphism. Define $\mathrm{Ker}\phi$, the **kernel** of the homomorphism.

(c) Given a homomorphism $\phi : G \to H$, prove that $\mathrm{Ker}\phi$ is a subgroup of G. (*Bonus:* Prove that $\mathrm{Ker}\phi$ is a *normal* subgroup of G.)

(4) (a) Define the **general linear group**, $GL_n(\mathbb{R})$, and the **special linear group**, $SL_n(\mathbb{R})$.

(b) Prove that $SL_2(\mathbb{R})$ is a subgroup of $GL_2(\mathbb{R})$.

(c) Assuming that $SL_n(\mathbb{R})$ is a subgroup of $GL_n(\mathbb{R})$ for all $n \geq 1$, prove that $SL_n(\mathbb{R})$ is a *normal* subgroup of $GL_n(\mathbb{R})$ by recognizing it as the kernel of a homomorphism $\phi : GL_n(\mathbb{R}) \to \mathbb{R}$.

(5) (a) State the **First Isomorphism Theorem** for groups.

(b) Let θ be a homomorphism from S_5 to S_2. Explain why $\mathrm{Im}\theta$ has either 1 or 2 elements.

(c) For the same homomorphism θ, explain why $K = \mathrm{Ker}\theta$ has index 1 or 2 in S_5 and conclude that K is either ________ or ________ (fill in the blanks and justify your answer).

(d) Deduce that there are only 2 homomorphisms from S_5 to S_2. Tabulate these two homomorphisms.

FINAL EXAM

TAKE-HOME PORTION

Please write or type your solutions to the following problems (on separate paper); they are due Tuesday, 12/11, at 10:30 (when our final exam begins). You may refer to your class notes and your book during this portion of the exam, *but you may not consult any other source of any type*. Problem 1 is worth 10 points; each of the other problems is worth 20 points.

Please initial that you agree to adhere to the conditions for this take-home exam: ____________.

(1) Imagine you are talking to a freshman math major; she is a sharp student who knows calculus and what the real numbers are, for example, but has no specific knowledge of group theory. Describe to her what a *group* is in a few sentences, and try to give her a sense of the general content and purpose of group theory. What's the goal of the subject? Why, as a math major, should she be required to take a course in it?
(Just a paragraph or two here.... I don't need a novel!)

(2) A group is **simple** if it has no nontrivial normal subgroups. Suppose G is a group of order 65. Use the Sylow Theorems to prove that G cannot be a simple group.

(3) Consider the group $G = \mathbb{Z}_4 \times \mathbb{Z}_6$.
   (a) What is $|G|$? List three elements in G and give their orders.
   (b) Let $H = \langle(0, 1)\rangle$ be the cyclic subgroup of G generated by the element $(0, 1)$. List the elements in H and explain how you know (without any computation) that H must be a normal subgroup of G.
   (c) How many (left) cosets does H have in G? List the elements in G/H by giving a representative for each coset.
   (d) Determine what group $(\mathbb{Z}_4 \times \mathbb{Z}_6)/\langle(0, 1)\rangle$ is isomorphic to and prove your answer.

(4) Let G be a group. Recall that Inn(G) = $\{\alpha_x \mid \alpha_x(g) = xgx^{-1}$ for all $g \in G\}$ is the set of all inner automorphisms of G. Define a map $\phi : G \to \text{Inn}(G)$ by $\phi(x) = \alpha_{x^{-1}}$.
   (a) Show that ϕ is a homomorphism.
   (b) Show that Imϕ =Inn(G) and Ker$\phi = \{x \in G \mid xg = gx$ for all $g \in G\}$.
   (Bonus point: the kernel is what familiar subgroup of G?)
   (c) When is ϕ an isomorphism? Be as specific as you can.

(5) EXTRA CREDIT
Let G be a group. Recall that the **center** of G is $Z(G) = \{x \in G \mid xg = gx$ for all $g \in G\}$, the set of elements x that commute with *all* elements in G.
   (a) Explain why $Z(G)$ is a normal subgroup of G. Explain why H is normal in G when H is any subgroup $H \leq Z(G)$.
   (b) If $H \leq Z(G)$ and G/H is cyclic, prove that G is Abelian.
   (Write down what it means for G/H to be cyclic.)

Bonus: Why is a carpool an Abelian group?

Math 3124 – Modern Algebra - 14674
Spring 2015
Tuesday/Thursday 12:30 - 1:45 (Newman Library 100)

Professor: Estrella Johnson
Email: strej@vt.edu
Office Hours: Mon/Thurs 10:00-11:00 and by appointment
Office: McBryde 444

Recommended Text: Abstract Algebra: Theory and Applications, Thomas Judson
http://abstract.ups.edu/download/aata-20140815.pdf

Course Objectives:

1. Develop your understanding of algebraic structures – particularly Groups, Rings, and Fields
2. Develop your ability to do mathematics, read mathematics, communicate mathematical ideas with others, make conjectures, solve problems, justify, and prove.

Course Format:
Class meetings will involve working in small groups on challenging problems, presenting your progress on these problems, and providing others with questions and comments. It is your responsibility as a learner and as a member of this class to be engaged in participation, rather than being just a passive observer or note-taker. As the instructor I will assume that when you stop discussing and asking questions, you no longer have any questions.

Homework:
Homework will be assigned most Thursdays and will be due the following Tuesdays. **Late assignments will not be accepted.** Turn in **something** for each problem to be collected; part of your grade will be assigned based on completion and part of it will be assigned based on the quality of your work (i.e. demonstrating your understanding of the concepts and presenting a clear, logical, convincing argument). You will be encouraged to work with others and use outside resources. **However, please remember to credit the people or sources that provided help with the different parts of your assignment.**

Portfolio:
You will be expected to keep a portfolio on the material covered in class. This portfolio will serve as your record of class activities and discussions. Additionally, you will be given "portfolio assignments" throughout the semester. These assignments will be designed to either wrap-up the work we did in class or to help prepare you for the next class session.

Your portfolio will be **graded 3 times** throughout the term. To receive full credit **you must be present** and have an entry for each class day and assignment. You are responsible for all portfolio entries, even for entries on days that you were absent.

Quizzes:
There will be two closed book quizzes. If you are going to miss a quiz (for a really good reason) you must make **prior arrangements** with me in order to make it up.

Exams:
Exam 1: Tentatively scheduled for Feb 19th
Exam 2: Tentatively scheduled for March 31st
Final: **Saturday, May 9th from 1:05-3:05**

> If you are going to miss an exam (for a really good reason) you must make **prior arrangements** with me in order to make it up. There will be no rescheduling or making up the final, regardless of the really good reason.

Course Grades:
Your final grade is determined using the following criterion:

Homework Assignments	15%
Portfolio	15%
Quizzes	10%
Exams and Final	60%

A scale no stricter than the standard grading scale will be used to assign letter grades.

Course Policies:

Attendance:
Attendance is expected for this class. Being absent is not an excuse to turn in assignments last or to have missing content in your portfolio. Remember **late homework assignments will not be accepted, you will receive a 0 for any portfolio check you miss,** and in order to make up quizzes/exams you must have **prior approval**.

Ethics and Integrity:
It is anticipated that students will benefit from discussions of course material with peers (or tutors) outside of class, outside resources, and even the Internet. However, the student is more likely to achieve the learning objectives of this course if his/her own work is submitted for analysis. By turning in a test or assignment, you (the student) certify that the work was produced without plagiarism (passing someone else's work as your own, or not citing someone else's work/contribution appropriately) or other forms of academic dishonesty. University policies concerning academic misconduct apply to this class and can be found at http://www.honorsystem.vt.edu

Makeup policy:
If you are registered for this course, it is expected that you will attend each day class is scheduled. There are absolutely no make-ups for missed homework. Exams can only be made up with prior arrangement with the instructor or legitimate excuse such as a doctor's note or funeral service card.

Students with Disabilities:
I wish to fully include any person with disabilities in this course. Any student who feels that he/she may need an accommodation because of a disability (learning disability, attention deficit disorder, psychological, physical, etc.), please make an appointment to see me.

3124 Spring 2014 Schedule

	Tuesday	Thursday
Week 1	**Jan 20th** Syllabus 1.1.1-1.1.8 Portfolio: 1.2.1	**Jan 22nd** 1.2.3-1.2.6 Portfolio: Finish 1.2.6 Homework: Prove there are only six symmetries, Prove that the combination of a symmetry is a symmetry, prove that any combination of 2 symmetries will be equivalent to one of our six symmetries
Week 2	**Jan 27th** 1.3.2-1.3.4 Portfolio: Start 1.3.5	**Jan 29th** 1.3.5 and 1.4.1 (Sudoku) Portfolio: 1.5.1 Homework: Right and Left Inverses and Inverses and Cancellation
Week 3	**Feb 3rd** Introduction to Permutations Portfolio:	**Feb 5th** 1.5.2 – 1.5.4 (Def of group) Portfolio: Are these a group? Homework:
Week 4	**Feb 10th** 1.6.1 – 1.6.3 and 1.7.2 Portfolio: 1.73 1.74	**Feb 12th** 1.8.0 – 1.8.3 Portfolio: 1.8.2 Homework: 1.8.4 – 1.8.6 and subgroups of D_8
Week 5	**Feb 17th** Review	**Feb 19th** Group Test
Week 6 Drop Day Mar 3rd	**Feb 24th** 2.1.1 2.2.2 Portfolio: 2.2.3	**Feb 26th** Functions and 2.2.3 (ISO Def) Homework: Getting used to def

Week 7	**Mar 3rd** 2.3 Portfolio:	**Mar 5th** 2.3 and ISO QUIZ
~~Week 8 SPRING BREAK~~	~~Mar 10th~~	~~Mar 12th~~
Week 9 Resign Mar 24th	**Mar 17th** QG lesson 1 Portfolio: Z_4	**Mar 19th** QG lesson 1 Homework
Week 10	**Mar 24th** QG lesson 2	**Mar 26th** QG lesson 2-3
Week 11	**Mar 31st** QG Lesson 3	**April 2nd** Homomorphisms
Week 12	**April 7th** Homomorphisms	**April 9th** FIT
Week 13	**April 14th** ISO and QG Exam	**April 16th** Intro to Rings
Week 14	**April 21nd** Def of Rings	**April 22th** Arithmetic Properties
Week 15	**April 28th** Subrings – Iso/homomor	**April 30th** Cancellation --- ID and Fields
Week 16 Classes end May 7th	**May 5th** Review without me	**May 7th** **Reading Day**
FINALS	SATURDAY MAY 10th 1:05-3:05	

Homework #3
Due Tuesday Feb 17th, In Class

Problem #1

Below you will find 6 conjectures about the minimal sufficient and necessary conditions needed to ensure that a subset of a group is a subgroup. For each conjecture, either prove that it is true or find a counterexample to show that it is false.

Conjecture 1:

Let ***G*** be a group and H a subset of ***G***. Then ***H*** is a subgroup iff 1) ***H*** is nonempty, 2) ***H*** is closed under the operation defined on ***G***, and 3) each element of ***H*** has its inverse in ***H***.

Conjecture 2:

Let ***G*** be a group and H a subset of ***G***. Then ***H*** is a subgroup iff 1) ***H*** is closed under the operation defined on ***G***, and 2) each element of ***H*** has its inverse in ***H***.

Conjecture 3:

Let ***G*** be a group and H a subset of ***G***. Then ***H*** is a subgroup iff 1) the identity of ***G***, e, is in ***H***, 2) ***H*** is closed under the operation defined on ***G***, and 3) each element of ***H*** has its inverse in ***H***.

Conjecture 4:

Let ***G*** be a group and H a subset of ***G***. Then ***H*** is a subgroup iff 1) the identity of ***G***, e, is in ***H***, 2) each element of ***H*** has its inverse in ***H***.

Conjecture 5:

Let ***G*** be a group and H a subset of ***G***. Then ***H*** is a subgroup iff H is closed.

Conjecture 6:

Let ***G*** be a group and H a subset of ***G***. Then ***H*** is a subgroup iff for all a,b in ***H***, ab^{-1} is also in ***H***.

Problem #2

Find all of the subgroups of D_8 (the symmetries of a square).

EXAM 1

SPRING 2014

PART A

Answer both of the following questions. Please justify your answers briefly but carefully.

(1) ***True or False?*** Decide if each statement is true or false, and give a brief justification or counterexample to support your answer.

(7.5 pts each)

(a) Suppose G is a group with $a^2 = \mathbf{e} = b^2$ and $ab = ba$, but $a \neq \mathbf{e}$, $b \neq \mathbf{e}$, and $a \neq b$. $H = \{\mathbf{e}, a, b, ab\}$ is a subgroup of G.

(b) If a nonempty subset H of a group G is closed under the operation of G, then H is a subgroup of G.

(c) Let a and b be elements in a group G, $ab^n a^{-1} = (aba^{-1})^n$.

(d) $\mathbb{Z}$ is a group under the operation of subtraction.

(2) Suppose that H and K are subgroups of a group G.

(a) Prove that the intersection $H \cap K$ is a subgroup of G.

(b) Find a counterexample that demonstrates the union $H \cup K$ is not necessarily a subgroup of G.

(20 pts)

PART B

Pick 2 of the following 4 questions and clearly indicate on your paper which problems you have chosen to answer (I will only grade 2 of your responses). Fully justified thinking and complete proofs are expected in each case. Each question is worth a total of 25 points.

(1) Suppose that G is a finite group.

  (a) Finish the following definition: "G is **commutative** if ..."

  (b) Provide an exmaple of a finite commutative group. Also, provide a *brief* explanation of how you now this is a commutative group.

  (c) Provide an exmaple of an infinte non-commutative group. Also, provide a *brief* explanation of how you now this is a non-commutative group.

  (d) Prove that if every non-identity element in G has order 2, then G is commutative.

(2) Let G be a group. The **center** of a group, G, is defined to be the subset consisting of all elements $c \in G$ such that $cx = xc$ for all $x \in G$ (*i.e.,* the elements that commute with everything in G). The center of G is usually denoted $Z(G)$.

  (a) Find the center of the symmetries of a square and justify your response.

  (b) Prove that $Z(G)$ is a subgroup of G

  (c) If $x \in Z(G)$ has order 4 and $y \in G$ has order 5, prove that xy has order 20.

(3) Let G be a group and let a and b be in G.

  (a) Prove that the order of a is the same as the order of a^{-1}. (*Hint, we know that* $a^{-n} = (a^{-1})^n$.)

  (b) Prove that the order of (ab) is the same as the order of (ba).

(4) Let G be the set of 2×2 real-valued matrices and let

$$H = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a + d = 0 \right\}.$$

  (a) Prove or disprove, H is a subgroup of G under addition.

  (b) Prove or disprove, H is a subgroup of G under multiplication.

Isomorphism, Quotient Groups, and Homomorphism Review
Test Tuesday, April 14th

Isomorphism

Be able to prove these things from class:

> Suppose $(G, *)$ and $(H, +)$ are isomorphic. If G is commutative then H is commutative.
>
> Suppose $(G, *)$ and $(H, +)$ are isomorphic and $\phi\colon G \longrightarrow H$ is an isomorphism. Let a be in G and n be a natural number, then $\phi(a^n) = \phi(a)^n$. [This should be a proof by mathematical induction.]
>
> Suppose $(G, *)$ and $(H, +)$ are isomorphic. If G is cyclic then H is cyclic.
>
> Suppose $(G, *)$ and $(H, +)$ are isomorphic and $\phi\colon G \longrightarrow H$ is an isomorphism. Let a be in G. Then $|\phi(a)| = |a|$.
>
> Suppose $(G, *)$ and $(H, +)$ are isomorphic and $\phi\colon G \longrightarrow H$ is an isomorphism. Then $\phi(e_G) = e_H$.

You might be asked to prove that two groups are or are not isomorphic. Examples:

> Prove from first principles (as opposed to simply citing a theorem) that $Z_4 \times Z_2$ cannot be isomorphic to Z_8.
>
> Let $G = \{e, a, a^2, a^3\}$ be a cyclic group of order 4 and H be the multiplicative group $\{1, -1, i, -i\}$ of complex numbers. Come up with 2 isomorphisms between G and H and explain why there are no others.
>
> Let $G = Z_2 \times Z_3$. Build the multiplication table for G. Is G isomorphic to D_6? Explain.

Quotient Groups

You may be given a familiar group (e.g., Z_n, D_8, D_6, $Z_m \times Z_n$) and asked to construct a quotient group of a certain size.

You may be asked to prove that, if H is a normal subgroup then G/H is a group.

You may have to find a normal subgroup or show that a given subgroup is or is not normal.

Homomorphism

You may be asked to come up with examples of homomorphisms (or asked to prove that such examples cannot exist): For instance, use $\boldsymbol{Z}_6$, and $\boldsymbol{D}_6$ to come up with each of the following (or prove why such a thing can not exist):

a. A homomorphism between two groups where one is commutative and the other is not.
b. A homomorphism where a self-inverse *is hit by* something other than a self-inverse.
c. A homomorphism where a self-inverse *is mapped to* something other than a self-inverse.
d. A homomorphism where an order 6 element maps to an order 3 element.
e. A homomorphism where an order 3 element maps to an order 6 element.

You might also be asked to prove things like:

If $\theta\colon G\to H$ is a homomorphism then $\theta(e_G) = e_H$ and $\theta(a^{-1}) = \theta(a)^{-1}$.

Fundamental (or First) Isomorphism (or Homomorphism) Theorem Questions

1) Prove that the image of a homomorphism is a subgroup
2) Prove that the kernel of a homomorphism is a subgroup
3) Prove that the kernel of a homomorphism is a normal subgroup
4) Given a homomorphism between two groups, construct a quotient group
5) Given a quotient group, construct a homomorphism
6) Given a homomorphism and a quotient group, show that the quotient group is isomorphic to the image of the homomorphism.

3124 Spring 2015
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(*20 points*) 1. Determine which of the following maps are homomorphisms and, for the ones that are homomorphisms, determine its kernel and image. (You do not need to prove your responses.)

a. $\theta: (\mathbb{Z}, +) \longrightarrow (\mathbb{R}, +)$ defined by $\theta(n) = n$.
b. $\theta: (\mathbb{R}, +) \longrightarrow (\mathbb{Z}, +)$ defined by $\theta(x) = the\ greatest\ integer\ \leq x$.
c. $\theta: \mathbb{Z}_6 \longrightarrow \mathbb{Z}_2$ defined by $\theta(n) = the\ remainder\ of\ n\ when\ divided\ by\ 2$.

(*20 points*) 2. Let $G = \{e, a, a^2, a^3\}$ be a cyclic group of order 4 and H be the multiplicative group $\{1, -1, i, -i\}$ of complex numbers, (where $i*i = -1$). Come up with 2 isomorphisms between G and H and explain why there are no others.

(*30 points*) 3. Commutativity

a. Suppose $\phi: G \longrightarrow H$ is an isomorphism. Prove that, if G is commutative then H is commutative.
b. Explain why your proof fails to show that, if there exists a homomorphism, $\theta: G \longrightarrow H$, and G is commutative then H is commutative.
c. Provide an example to show that it is possible for there to exist a homomorphism, $\theta: G \longrightarrow H$, where G is commutative and H is not.

(*30 points*) 4. Let $\mathbb{Z}_7^*$ be the set of numbers in $\mathbb{Z}_7$ that have a ***multiplicative inverse***. For instance, 6*6 = 36 ≡ 1 (mod 7). So, because 6 has a multiplicative inverse, is it in the set $\mathbb{Z}_7^*$. Below is an operation table for ($\mathbb{Z}_7^*$,×).

×	**1**	**2**	**3**	**4**	**5**	**6**
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

a. Form a 3-element quotient group out of ($\mathbb{Z}_7^*$,×) and make an operation table of this new group.
b. Construct a homomorphism, θ, from ($\mathbb{Z}_7^*$,×) to $\mathbb{Z}_6$ such that the *ker*θ has two elements.
c. Show that the quotient group you constructed in part a. is isomorphic to the image of $\mathbb{Z}_7^*$ under the θ you constructed in part b.

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(*20 points*) 1. Determine if the following are *always, sometimes,* or *never* true (you don't need to justify your answer):

a. Let $a, b \in G$ where G is a group, $(ab)^2 = a^2b^2$
b. Let R be a ring with a, $b \in R$, $-(a + b) = -a + -b$.
c. Let R be a ring with a, b, $c \in R$ not equal to 0. If $a(b - c) = 0$, then $b = c$.
d. Let H be a subgroup of G. If $x \notin H$ and $y \notin H$, then $xy \notin H$.

(*15 points*) 2. Consider the operation tables for the following ring:

+	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
0	0	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8	0
2	2	3	4	5	6	7	8	0	1
3	3	4	5	6	7	8	0	1	2
4	4	5	6	7	8	0	1	2	3
5	5	6	7	8	0	1	2	3	4
6	6	7	8	0	1	2	3	4	5
7	7	8	0	1	2	3	4	5	6
8	8	0	1	2	3	4	5	6	7

*	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8
2	0	2	4	6	8	1	3	5	7
3	0	3	6	0	3	6	0	3	6
4	0	4	8	3	7	2	6	1	5
5	0	5	1	6	2	7	3	8	4
6	0	6	3	0	6	3	0	6	3
7	0	7	5	3	1	8	6	4	2
8	0	8	7	6	5	4	3	2	1

a. What sort of ring is this (e.g., ring, integral domain, …)? Be as specific as possible and justify your answer.
b. Find a non-trivial subring (i.e. not just {0}). What sort of ring is the subring you found? Be as specific as possible and justify your answer.
c. Is it possible to find a subring of this ring that is itself a field? Justify your answer.

(*15 points*) 3. Suppose that the group G has order 12, that φ is a homomorphism from G to H, and that G has 3 normal subgroups: $\{0\}$, all of G, and K, a three element subgroup. Use the fundamental homomorphism theorem to determine the possible orders of $\varphi(G)$. Explain your solution.

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(*20 points*) 4. Identity and Unity

a. Let $(G, +)$ be a group and let $(H, +)$ be a subgroup of G. Prove that the identity of G must be the same as the identity of H.
b. Let $(G, +)$ be a group and let $(H, +)$ be a subgroup of G and let h be in H. Prove the inverse of h in H is the same as the inverse of h in G.
c. Explain why your proof in a. fails to show that, if $(G, +, *)$ is a ring with unity and $(H, +, *)$ is a subring with unity of G, then the unity of G must be the same as the unity of H.
d. Provide an example to show that it is possible for $(G, +, *)$ and $(H, +, *)$ to have different unities.

(*30 points*) 5. Kernel of a Homomorphism (Suppose G and H are groups)

a. Give the definition of a **homomorphism** from G to H.
b. Suppose $\varphi : G \to H$ is a homomorphism. Define Kerφ, the **kernel** of the homomorphism.
c. Given a homomorphism $\varphi : G \to H$, prove that Kerφ is a subgroup of G (prove *everything* you need to show that this is a subgroup).
d. Prove that Kerφ is a *normal* subgroup of G (if you get stuck – try a different version of the normality definition).

Permutations

Let's say we were working with the following set of numbers {1,2,3} and I wanted to keep track of the different functions from {1,2,3} to {1,2,3}. Even better, let's only keep track of functions that are one-to-one and onto.

So, I want to include functions like the following two:

$1 \to 2$	$1 \to 1$
$2 \to 3$	$2 \to 3$
$3 \to 1$	$3 \to 2$

But I don't want to count things like either of the following:

$1 \to 1$	$1 \to 3$
$2 \to 1$	$2 \to 3$
$3 \to 1$	$3 \to 2$

Cauchy was considering such functions and he got tired of writing arrows. So he developed a slightly shorter "double decker" notation. On the top row is the list of elements, and on the bottom row we keep track of where those elements go.

For instance, Cauchy would represent

$$\begin{matrix} 1 \to 2 \\ 2 \to 3 \\ 3 \to 1 \end{matrix} \qquad \text{as} \qquad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

Not too surprisingly (since these things are functions) we can compose them. So, let's say that we wanted to combine two of these functions, and then see what we get out:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

Did you see what happened there? Be careful, the notation is a little off because it is function composition, so we do the function on the right first, and then do the function on the left.

So with $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$

> The function on the right sends $1 \to 1$ and then the function on the left send $1 \to 2$. So, all together, $1 \to 2$.
>
> Then the first function sends $2 \to 3$ and the second sends $3 \to 1$, so all together $2 \to 1$.
>
> Finally, $3 \to 2$ and then $2 \to 3$, so all together $3 \to 3$.

Your job:

1. Identify all of the bijective functions from {1,2,3} to {1,2,3}.
2. Notate each of those functions using Cauchy's double decker notation
3. Make a table showing the results of combining all pairs of functions (still doing the side first and then the top).
4. Make a list of "official rules" for these calculations.