

## A Rate of Change Equation for Limited Resources

In a previous problem we saw that the rate of change equation  $\frac{dP}{dt} = 0.3P$  can be used to model a situation where there is one species, continuous reproduction, and unlimited resources. In most situations, however, the resources are not unlimited, so to improve the model one has to modify the rate of change equation  $\frac{dP}{dt} = 0.3P$  to account for the fact that resources are limited.

1. (a) In what ways does the modified rate of change equation

$$\frac{dP}{dt} = 0.3P \left(1 - \frac{P}{10}\right)$$

account for limited resources? (Think of 10 as scaled to mean 10,000 or 100,000)

- (b) How do you interpret the solution with initial condition  $P(0) = 10$ ?

- (c) Open the Slope Field Viewer, <https://ggbm.at/ZGeeGQbp>, and plot the slope field for



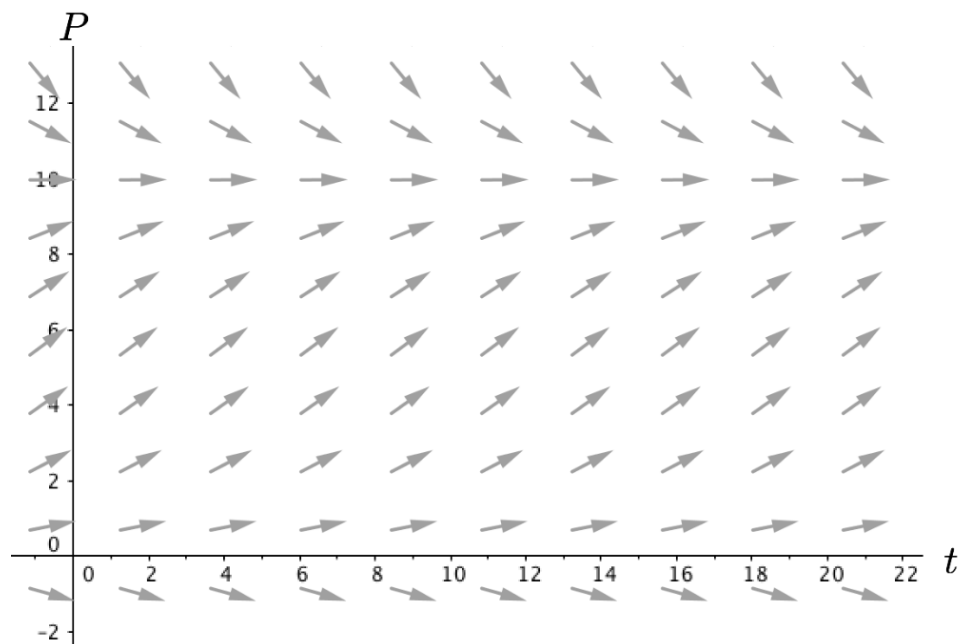
$$\frac{dP}{dt} = 0.3P \left(1 - \frac{P}{10}\right).$$

(Note: In the Slope Field Viewer you will need to use the variable  $y$  instead of  $P$ , and you may want to change the viewing window using the button on the right of the applet.) In what ways are your responses to parts 1a and 1b visible in the slope field?

- (d) In this problem, negative  $P$  values do not make sense, but we can still mathematically make sense of the slope field for negative  $P$  values. Explain why the slope field looks the way it does below the  $t$ -axis.
- (e) If there are initially  $P(0) = 2$  fish in the lake, approximately how many fish are in the lake at time  $t = 2$ ? How did you arrive at your approximation?

## Using a Slope Field to Predict Future Fish Populations

Below is a slope field for the rate of change equation  $\frac{dP}{dt} = 0.3P \left(1 - \frac{P}{10}\right)$ .



2. (a) On the slope field above, stitch together in a tip to tail manner several tangent vectors to produce a graph of the population versus time if at time  $t = 0$  we know there are 5 fish in the lake (again, think of 5 as scaled for say, 5000 or 50,000).
- (b) Reproduce your technique as much as possible using the Slope Field Stitcher applet, <https://ggbm.at/FZn4WHeU>. You can use the arrow buttons to move the initial vector around, and then create subsequent vectors to stitch on using the appropriate button.



3. Explain how you are thinking about rate of change **in your method**. For example, is the rate of change constant over some increment? If yes, over what increment? If no, is the rate of change always changing?

4. Using the differential equation  $\frac{dP}{dt} = P \left( 1 - \frac{P}{20} \right)$  and initial condition  $P(0) = 10$ , José and Julie started the following table to numerically keep track of their tip-to-tail method for connecting tangent vectors. Explain José's and Julie's approach and complete their table. Round to two decimal places.

$t$	$P$	$\frac{dP}{dt}$
0	10	5
0.5	12.5	
1.0		
1.5		

5. Using the same differential equation and initial condition as José and Julie, Derrick and Delores started their table as shown below. Explain how Derrick and Delores' approach is different from José and Julie's and then complete their table. Round to two decimal places.

$$\frac{dP}{dt} = P \left( 1 - \frac{P}{20} \right)$$

$t$	$P$	$\frac{dP}{dt}$
0	10	5
.25	11.25	
.5		
.75		

6. Which approach do you think is more accurate and why?

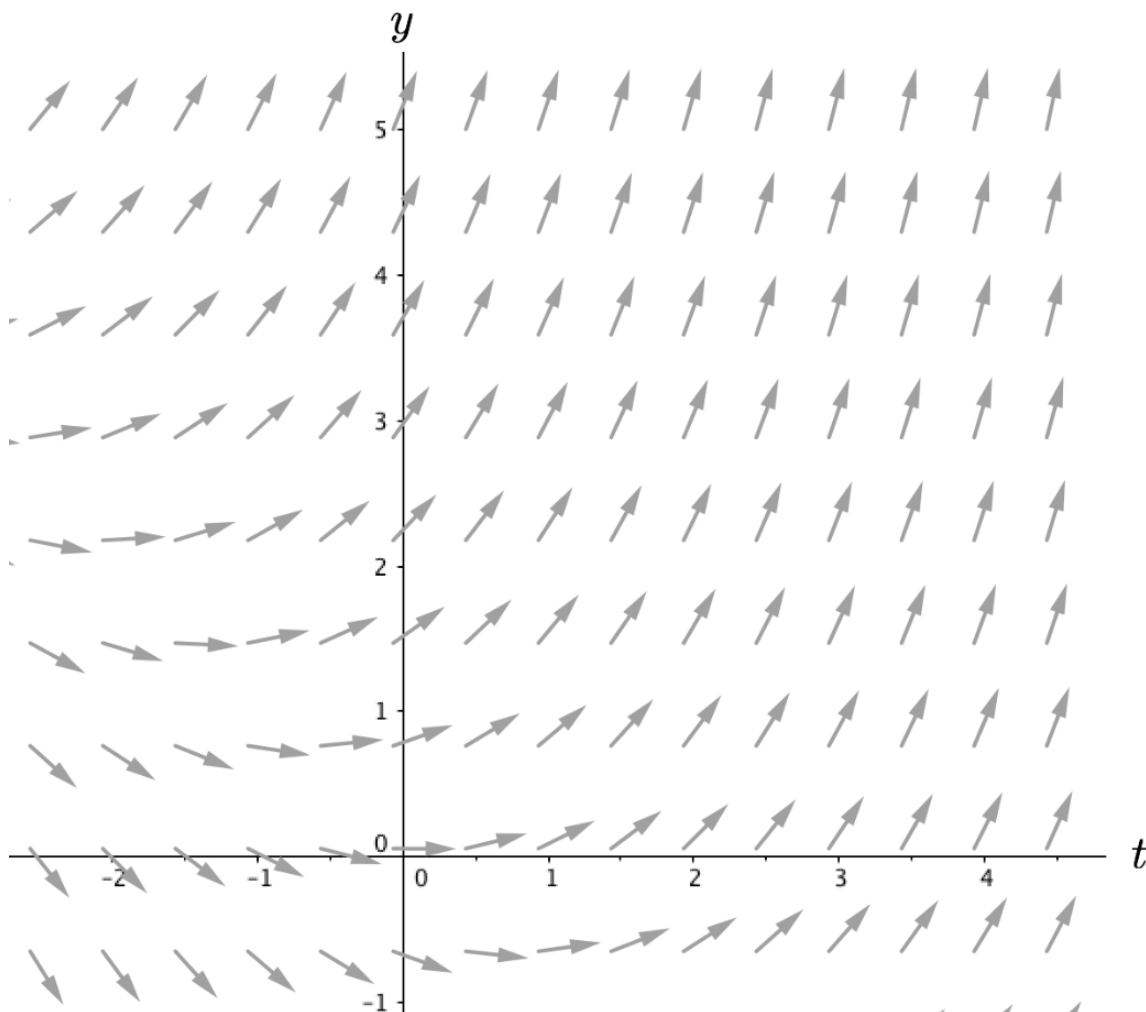
7. (a) Consider the differential equation  $\frac{dy}{dt} = y + t$  and initial condition  $y(0) = 4$ . Use José and Julie's approach to find  $y(1.5)$ . Show your work graphically and in a table of values.

(b) Is your value for  $y(1.5)$  the exact value or an approximate value? Explain.

8. **Generalizing your tip-to-tail approach.** Create an equation-based procedure/algorithm that would allow you to predict future  $y$ -values for any differential equation  $\frac{dy}{dt}$ , any given initial condition, and any time increment.

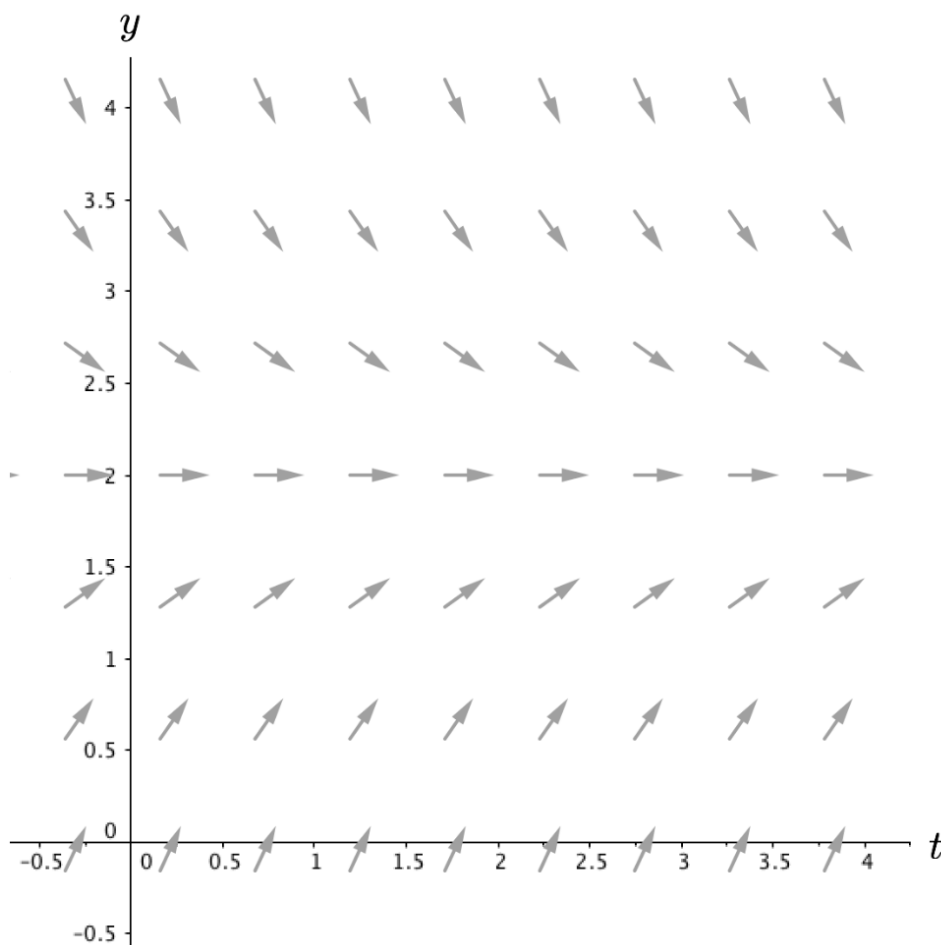
## Homework Set 2

1. A slope field for the differential equation  $\frac{dy}{dt} = 0.5(y + t)$  is shown below.



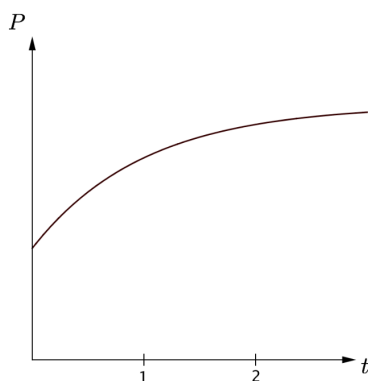
- For the initial condition of  $y(0) = 1$ , sketch on the above slope field what you think two iterations of the “tip-to-tail” method with a step size of 1 unit should look like. Do this **without** doing any computations.
- Again, without doing any computations, sketch on the same slope field what you think three iterations with a step size of 0.5 units should look like for the same initial condition (perhaps using a different color).
- Use the tip-to-tail (*i.e.*, Euler’s) method to numerically compute approximations for parts 1a and 1b and then compare your graphical predictions to the numerical results.

2. Consider a differential equation with the given slope field and the initial value  $y(0) = 1$ .

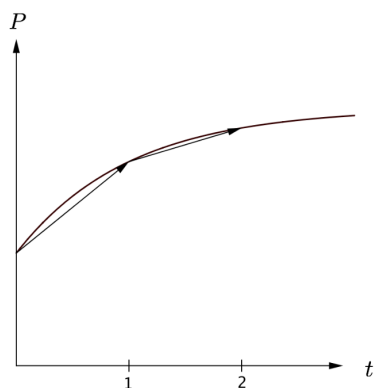


- Explain why, if you wanted to approximate  $y(2)$  using two steps of Euler's method, you would need  $\Delta t = 1$ .
- Use a straight edge to graph two steps of Euler's method to approximate  $y(2)$ .
- This time, instead of using two steps of Euler's method, sketch on the same slope field what it would look like if you used four steps of Euler's method to approximate  $y(2)$ .
- Besides the obvious difference that the step size is different, state two other things that are different between your answers to parts 2b and 2c.
- Besides the obvious fact that they both use Euler's method, what is similar about the first step to your answers to parts 2b and 2c?

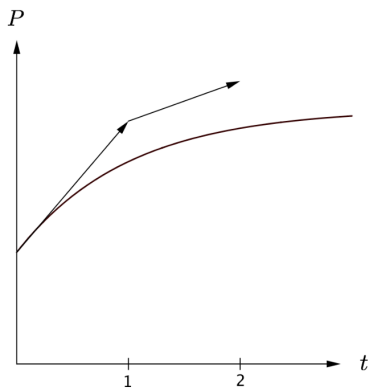
3. Suppose we have a rate of change equation and initial condition for the population of raccoons in Lake County. Below is a graph of an **exact** solution.



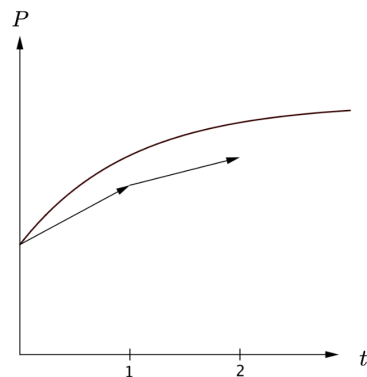
Merry, Pippin, and Sam attempted to use the “tip-to-tail” Euler method to predict what the population of raccoons would be at time  $t = 2$ , with time increments one unit. However, they arrived at different graphs for their predictions. Their predictions are given below, and are shown with the exact solution.



Merry's Prediction



Pippin's Prediction



Sam's Prediction

For each prediction, give reasons as to whether or not each person illustrated the correct relationship between Euler's method and the exact solution.

4. Suppose the function  $y(t) = 6t + 1$  is a solution to a particular differential equation. For the initial condition  $y(0) = 1$ , is a graph of the tip-to-tail Euler method exactly the same as the graph of the exact solution? Does your response depend on step size? Explain.
5. Compute by hand four steps of the tip-to-tail Euler method for the differential equation  $\frac{dy}{dt} = y - t$  with initial condition  $y(0) = 2$  and step size 0.5.

6. **Euler's Method Using a Spreadsheet.** Learning to use a spread sheet for various applications in engineering and mathematics is a valuable skill. Your task in this problem is to use Excel to generate as many steps of the Euler method that you want. *If you are already familiar with Excel, skip the example below and go directly to part a.*

EXAMPLE: Here are step by step instructions for how to use Excel to generate 15 steps of the algorithm  $Y_{\text{next}} = 2 \cdot Y_{\text{now}} + 1$  with initial condition  $Y = 3$ .

- Open an Excel workbook
  - Select cell A1 by clicking on the cell in this location and type in Ynow as a column heading
  - Select cell B1 and create a column heading called Ynext
  - Select cell A2 and type in the number 3 (this is the given initial Y-value)
  - Select cell B2 and type =2\*A2+1 (after pressing Enter the number 7 will appear in this cell)
  - Select cell A3 and type =B2
  - Select and copy cell B2 (An animated dashed-line will appear around the cell)
  - Select cells B3 through B15 and paste
  - Select and copy cell A3
  - Select cells A4 through A15 and paste
  - Do a few hand computations to verify the results
- (a) Using a step size  $\Delta t$  of your choice, figure out how to use Excel to generate at least 20 steps for Euler's method,  $y_{\text{next}} = y_{\text{now}} + \left(\frac{dy}{dt}\right)_{\text{now}} \cdot \Delta t$ , for the differential equation  $\frac{dy}{dt} = 0.3y(1 - \frac{y}{12.5})$  with initial condition  $y(0) = 3$ . In order to make it easier to graph the results, make your first column  $t_{\text{now}}$  and your second column  $y_{\text{now}}$ . Turn in a print out your results and verify the first three steps by hand.
- (b) Use the Chart Wizard scatter plot option to create a graph of your  $(t, y)$  data from part 6a. An easy way to do this is to first highlight all the data in the  $t_{\text{now}}$  and  $y_{\text{now}}$  columns, select Chart Wizard, and follow the prompts. Turn in a print out of your results.
7. Two students are having a discussion about the equal sign in the rate of change equation  $\frac{dP}{dt} = 0.5P \left(1 - \frac{P}{100}\right)$ . One student says he thinks about the equal sign as instructions for calculating. The other student says he thinks about the equal sign as a kind of mirror. How do you think about the equal sign in a rate of change equation?



8. A group of scientists created the differential equation  $\frac{dP}{dt} = 0.8P \left(1 - \frac{P}{5}\right)$  to predict future fish populations in Lake Minnetonka, where  $P$  represents thousands of fish and  $t$  is in years.
- If you were to plot a slope field for this rate of change equation, what window for the  $P$  and  $t$  values would you use to make sure the most important features are clearly shown? Explain.
  - What does this rate of change equation predict about the long-term outcome of the fish population if the initial population is 2 (*i.e.*,  $P = 2$  at  $t = 0$ )? How about if  $P = 6$  at  $t = 0$ ?
  - Why are the predictions you made in part 8b reasonable (or not) for a fish population? Explain.
  - Carry out by hand three steps of Euler's method with a step size of 0.5 for the initial condition  $P(0) = 5$ .